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INHOMOGENEOUS QUANTUM GROUPS $IGL_{q,r}(N)$: UNIVERSAL ENVELOPING ALGEBRA AND DIFFERENTIAL CALCULUS

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Abstract

A review of the multiparametric linear quantum group $GL_{q,r}(N)$, its real forms, its dual algebra $U(gl_{q,r}(N))$ and its bicovariant differential calculus is given in the first part of the paper.

We then construct the (multiparametric) linear inhomogeneous quantum group $IGL_{q,r}(N)$ as a projection from $GL_{q,r}(N+1)$, or equivalently, as a quotient of $GL_{q,r}(N+1)$ with respect to a suitable Hopf algebra ideal.

A bicovariant differential calculus on $IGL_{q,r}(N)$ is explicitly obtained as a projection from the one on $GL_{q,r}(N+1)$.

Our procedure unifies in a single structure the quantum plane coordinates and the q-group matrix elements $T^a{}_b$, and allows to deduce without effort the differential calculus on the q-plane $IGL_{q,r}(N)/GL_{q,r}(N)$.

The general theory is illustrated on the example of $IGL_{q,r}(2)$.

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1 Introduction

Quantum deformations of inhomogeneous Lie groups have been studied recently [1]. An R-matrix approach has been independently proposed for $IGL_q(N)$ in ref.s [2] and [3] ; in [2] the corresponding universal enveloping algebra is discussed, while in [3] a bicovariant differential calculus for $IGL_q(N)$ is obtained. In ref. [4] the $IGL_q(N)$ Hopf algebra and its bicovariant differential calculus are obtained via a projection from $GL_q(N+1)$. The same idea was used in ref.s [5] to find the bicovariant calculus on the inhomogeneous q-groups of the B, C, D type, and a q-deformation of the Einstein-Cartan lagrangian of gravity based on $ISO_q(3,1)$.

In this paper we construct the *multiparametric* $IGL_{q,r}(N)$ quantum groups and their bicovariant differential calculus by using the projective method of [4, 5]. As we found already in [5], for inhomogeneous q-groups it is essential to consider the most general (multiparametric) case. Indeed we find that only in some of these deformations the dilatation part can be set to the identity.

All the quantities relevant to their (bicovariant) differential calculus are given explicitly: exterior derivatives, left-invariant one-forms, Cartan-Maurer equations, tangent vectors and their q-Lie algebra and so on. The method is illustrated in the case of $IGL_{q,r}(2)$: the general formulas are applied and tested on this example (see the Table).

Our framework allows us to construct the differential geometry of the (multiparametric) quantum plane in a novel and easy way. We obtain a generalization of the q-plane of ref. [6]; this last is recovered when setting all parameters equal to a single parameter q.

In Section 2 we recall the basics of the linear quantum groups (see ref.s [7] - [10]), and in Section 3 we discuss their duals in some detail. In fact, Sections 2 and 3 are a short review of the multiparametric deformations of $GL_{q,r}(N)$, where q indicates a set of parameters q_i, and of their universal enveloping algebras. The usual uniparametric case is recovered for $r = q_i = q$. For references on multiparametric deformations, see [11, 12].

The explicit construction of the bicovariant differential calculus for $GL_{q,r}(N)$, in terms of the dual algebra, is given in Section 4. Some new results include an inversion formula for the left-invariant one-forms in terms of derivatives of the group elements (the q-analogue of $\omega = g^{-1}dg$). For a review of the differential geometry on quantum groups see for ex. [10]. This subject, initiated in [13], has been actively developed in recent years: an incomplete list of references can be found in [14] - [19].

In Section 5 we first present the quantum group $IGL_{q,r}(N)$ as a Hopf algebra with given generators, commutation relations and co-structures. We then reobtain it as the image of a projection P from $GL_{q,r}(N+1)$, and show how the "mother" Hopf algebra of $GL_{q,r}(N+1)$ determines the Hopf algebra structure on $IGL_{q,r}(N)$. In the

language of Hopf algebra ideals $IGL_{q,r}(N)$ is seen as the quotient of $GL_{q,r}(N+1)$ with respect to a suitable Hopf ideal.

The fundamental representation of $IGL_{q,r}(N)$ contains the $GL_{q,r}(N)$ elements $T^a{}_b$ and the "coordinates" x^a, as in the classical case; in addition, there is also an element u playing the role of a dilatation. By fixing some of the parameters q, we find that this element u can be made central, and hence consistently set equal to the identity I.

A quantum determinant can be defined, and is central only in a subclass of the multiparametric deformations. In this subclass, however, the element u is not central.

In Section 6 we project the bicovariant differential calculus of $GL_{q,r}(N+1)$ to $IGL_{q,r}(N)$ and study the bicovariant bimodules of 1-forms and tangent vectors on $IGL_{q,r}(N)$. In particular, the q-Lie algebra is given explicitly. We also study in detail the exterior algebra and the exterior derivative, and find the Cartan-Maurer equations.

In Section 7 we discuss the multiparametric quantum plane, i.e. the quantum coset space $IGL_{q,r}(N)/GL_{q,r}(N)$ spanned by the coordinates x^a, and find a generalization of the differential geometry of the q-plane of [6] (see also Schirrmacher in [12]).

In the Table at the end of the paper we specialize our general treatment to $IGL_{q,r}(2)$ and collect all the relevant formulas for its bicovariant differential calculus.

2 $GL_{q,r}(N)$ and its real forms

First, we recall the definition of $GL_{q,r}(N)$. It is the algebra (over the complex field) freely generated by the non-commuting matrix elements $T^A{}_B$, (A,B=1,..N), the identity I and the inverse Ξ of the q-determinant of T defined in (2.7), modulo the "RTT" relations:

$$R^{AB}{}_{EF}T^E{}_CT^F{}_D = T^B{}_FT^A{}_ER^{EF}{}_{CD} \tag{2.1}$$

where the R-matrix is given by [11]:

$$R^{AB}{}_{CD} = \delta^A_C\delta^B_D[\frac{r}{q_{AB}} + (r-1)\delta^{AB}] + (r-r^{-1})\ \delta^A_D\delta^B_C\theta^{AB} \tag{2.2}$$

with $\theta^{AB} = 1$ for $A > B$ and zero otherwise, and

$$q_{AB} = \frac{r^2}{q_{BA}}, \quad q_{AA} = r \tag{2.3}$$

It is useful to list the nonzero complex components of the R matrix (no sum on repeated indices):

$$R^{AA}{}_{AA} = r$$

$$
\begin{aligned}
R^{AB}{}_{AB} &= \frac{r}{q_{AB}}, \qquad A \neq B \\
R^{BA}{}_{AB} &= r - r^{-1}, \quad B > A
\end{aligned}
\tag{2.4}
$$

The R matrix in (2.2) satisfies the quantum Yang-Baxter (QYB) equation:

$$
R^{A_1B_1}{}_{A_2B_2} R^{A_2C_1}{}_{A_3C_2} R^{B_2C_2}{}_{B_3C_3} = R^{B_1C_1}{}_{B_2C_2} R^{A_1C_2}{}_{A_2C_3} R^{A_2B_2}{}_{A_3B_3}. \tag{2.5}
$$

The standard uniparametric R matrix [8] is obtained from (2.2) by setting all deformation parameters q_{AB}, r equal to a single parameter q.

The quantum determinant of T and its inverse Ξ are defined by:

$$
\Xi \ \mathrm{det} T = \mathrm{det} T \ \Xi = I \tag{2.6}
$$

$$
\mathrm{det} T \equiv \sum_\sigma \left[\prod_{A<B, \sigma(A)>\sigma(B)} \left(-\frac{r^2}{q_{\sigma(B)\sigma(A)}}\right) \right] T^1{}_{\sigma(1)} \cdots T^N{}_{\sigma(N)} \tag{2.7}
$$

Note 1: In the uniparametric case $r = q_{AB} = q$ we recover the usual formula

$$
\mathrm{det} T \equiv \sum_\sigma (-q)^{l(\sigma)} T^1{}_{\sigma(1)} \cdots T^N{}_{\sigma(N)} \tag{2.8}
$$

where $l(\sigma)$ is the minimum number of transpositions in the permutation σ.

Note 2: In more mathematical terms, the algebra $GL_{q,r}(N)$ is the quotient of the non-commuting algebra $\mathbf{C}\langle T^A{}_B, I, \Xi \rangle$ freely generated by the elements $T^A{}_B, I, \Xi$ with respect to the two-sided ideal in $\mathbf{C}\langle T^A{}_B, I, \Xi \rangle$ generated by the RTT relations (2.1).

Note 3: the inverse matrix R^{-1}, defined as

$$
(R^{-1})^{AB}{}_{CD} R^{CD}{}_{EF} \equiv \delta^A_E \delta^B_F \equiv R^{AB}{}_{CD} (R^{-1})^{CD}{}_{EF}. \tag{2.9}
$$

is given by

$$
R^{-1}_{q,r} = R_{q^{-1},r^{-1}} \tag{2.10}
$$

Note 4: the $\hat{R}$ matrix defined by $\hat{R}^{AB}{}_{CD} \equiv R^{BA}{}_{CD}$ satisfies the spectral decomposition (Hecke condition):

$$
(\hat{R} - rI)\ (\hat{R} + r^{-1}I) = 0 \tag{2.11}
$$

Note 5: the determinant in (2.7) is central if and only if the following conditions on the parameters are satisfied (see ref. [11]):

$$
q_{1,A} q_{2,A} \cdots q_{A-1,A} \frac{r^2}{q_{A,A+1}} \frac{r^2}{q_{A,A+2}} \cdots \frac{r^2}{q_{A,N}} = const. \tag{2.12}
$$

for all A=1,...N. This results in N-1 conditions among the q_{AB} and determines $const = r^{N-1}$. Using (2.3), and defining

$$Q_A \equiv \prod_{C=1}^{N} (\frac{q_{CA}}{r}) \tag{2.13}$$

the centrality conditions (2.12) become:

$$Q_A = 1 \tag{2.14}$$

We have used also $const = r^{N-1}$, so that only $N-1$ of the above conditions are independent. Indeed the Q_A satisfy the relation

$$Q_1 Q_2 \cdots Q_N = 1 \tag{2.15}$$

In general we have:

$$(\det T) T^A{}_B = \frac{Q_A}{Q_B} T^A{}_B (\det T), \quad \Xi T^A{}_B = \frac{Q_B}{Q_A} T^A{}_B \Xi \tag{2.16}$$

When (2.14) holds, we can consistently set $\det T^A{}_B = I = \Xi$, and obtain the multiparametric deformations $SL_{q,r}(N)$.

The RTT equation (2.1) and the quantum Yang–Baxter equation can be rewritten more compactly as:

$$R_{12} T_1 T_2 = T_2 T_1 R_{12} \tag{2.17}$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \tag{2.18}$$

where the subscripts 1, 2 and 3 refer to different couples of indices. Thus T_1 indicates the matrix $T^A{}_B$, $T_1 T_1$ indicates $T^A{}_C T^C{}_B$, $R_{12} T_2$ indicates $R^{AB}{}_{CD} T^D{}_E$ and so on, repeated subscripts meaning matrix multiplication.

The algebra $GL_{q,r}(N)$ becomes a Hopf algebra with the following coproduct Δ, counit ε and coinverse κ:

$$\Delta(T^A{}_B) = T^A{}_B \otimes T^B{}_C \tag{2.19}$$

$$\varepsilon(T^A{}_B) = \delta^A_B \tag{2.20}$$

$$\kappa(T^A{}_B) = (T^{-1})^A{}_B \tag{2.21}$$

$$\Delta(\det T) = \det T \otimes \det T, \quad \Delta(\Xi) = \Xi \otimes \Xi, \quad \Delta(I) = I \otimes I \tag{2.22}$$

$$\varepsilon(\det T) = 1, \quad \varepsilon(\Xi) = 1, \quad \varepsilon(I) = 1 \tag{2.23}$$

$$\kappa(\det T) = \Xi, \quad \kappa(\Xi) = \det T, \quad \kappa(I) = I \tag{2.24}$$

The quantum inverse of $T^A{}_B$ in (2.21) is given by:

$$(T^{-1})^A{}_B = \Xi \ \Pi^{(1,N)}_{AB} \ t^{\ A}_B \tag{2.25}$$

where t^A_B is the quantum minor, i.e. the quantum determinant of the submatrix of T obtained by removing the B-th row and the A-th column, and $\Pi^{(1,N)}_{AB}$ is a function of the parameters q:

$$\Pi^{(1,N)}_{AB} \equiv \frac{\prod^N_{C=B+1}(-q_{BC})}{\prod^N_{D=A+1}(-q_{AD})} \tag{2.26}$$

The superscript (1,N) reminds the range of the indices A,B,C,... In the uniparametric case, the quantum inverse has the simpler expression:

$$(T^{-1})^A{}_B = \Xi\ (-q)^{A-B} t^A_B \tag{2.27}$$

Note 5: In general $\kappa^2 \neq 1$. The following useful relation holds

$$\kappa^2(T^A{}_B) = D^A_C T^C{}_D (D^{-1})^D_B = d^A d_B^{-1} T^A{}_B, \tag{2.28}$$

where D is a diagonal matrix, $D^A_B = d^A \delta^A_B$, given by $d^A = r^{2A-1}$ for $GL_{q,r}(N)$. This matrix satisfies:

$$d^A d_C^{-1} (R^{-1})^{BA}{}_{DC} R^{EC}{}_{BF} = \delta^A_F \delta^E_D, \quad d^A d_C^{-1} R^{AB}{}_{CD} (R^{-1})^{CE}{}_{FB} = \delta^A_F \delta^E_D \tag{2.29}$$

$$d^B d_D^{-1} (R^{-1})^{AB}{}_{CD} R^{CE}{}_{FB} = \delta^A_F \delta^E_D, \quad d^B d_D^{-1} R^{BA}{}_{DC} (R^{-1})^{EC}{}_{BF} = \delta^A_F \delta^E_D \tag{2.30}$$

$$R^{AC}{}_{CB} d_C^{-1} = \delta^A_B = (R^{-1})^{AC}{}_{CB} d_C \tag{2.31}$$

Relations (2.29) and (2.30) define a second inverse $R^{\sim 1}$ of the R matrix and a second inverse $(R^{-1})^{\sim 1}$ of the R^{-1} matrix as:

$$(R^{\sim 1})^{AB}{}_{CD} \equiv d^B d_D^{-1} (R^{-1})^{AB}{}_{CD} \tag{2.32}$$

$$((R^{-1})^{\sim 1})^{AB}{}_{CD} \equiv d^A d_C^{-1} R^{AB}{}_{CD} \tag{2.33}$$

Using (2.31) we can relate the D matrix to this second inverse:

$$(D^{-1})^A_B = (R^{\sim 1})^{AC}{}_{CB}, \quad D^A_B = ((R^{-1})^{\sim 1})^{AC}{}_{CB} \tag{2.34}$$

This generalizes the analogous discussion for the uniparametric D matrix given in [8].

We turn now to the real forms of $GL_{q,r}(N)$. These are defined by *-involutions of the $GL_{q,r}(N)$ Hopf algebra, that is mappings which are algebra antimorphisms and a co-algebra automorphisms:

$$(\lambda a)^* = \bar{\lambda} a^*, \quad (ab)^* = b^* a^*, \quad \Delta(a^*) = [\Delta(a)]^* \quad \lambda \in \mathbf{C}; \quad a, b \in GL_{q,r}(N) \tag{2.35}$$

($\bar{\lambda}$ is the usual complex conjugate of λ) and satisfy the involution conditions:

$$(a^*)^* = a, \quad \kappa([\kappa(a^*)]^*) = a \quad \forall a \in GL_{q,r}(N) \tag{2.36}$$

Moreover, these involutions (also called conjugations) must be compatible with the RTT relations: this restricts the range of the parameters q, r. Three such conjugations are known (cf. [11]):

i) $T^* = T$, i.e. the elements $T^A{}_B$ are "real". Applying the *-conjugation to the RTT equations (2.1) yields again the RTT relations if the R matrix satisfies $\bar{R} = R^{-1}$. This happens for $|q_{AB}| = |r| = 1$, i.e. for deformation parameters lying on the unit circle in $\mathbf{C}$ (cf. eq. (2.10)). The quantum group is then denoted by $GL_{q,r}(N;\mathbf{R})$.

ii) $(T^A{}_B)^* = T^{A'}{}_{B'}$ with primed indices defined as $A' = N+1-A$. Here compatibility with the RTT relations requires $\bar{R}^{AB}{}_{CD} = R^{B'A'}{}_{D'C'}$, satisfied when $\bar{q}_{AB} = q_{B'A'},\ r \in \mathbf{R}$. We can then define "real" generators as $(T+T^*)/2$, and the corresponding quantum group could be called $GL'_{q,r}(N;\mathbf{R})$.

iii) $(T^A{}_B)^* = \kappa(T^B{}_A)$, the generalization of the unitarity condition for the matrix T. In this case (left as an exercise in [11]) the restriction on the R matrix is $\bar{R}^{AB}{}_{CD} = R^{DC}{}_{BA}$, leading to the conditions $\bar{q}_{AB} = q_{BA},\ r \in \mathbf{R}$. The corresponding quantum groups are denoted by $U_{q,r}(N)$.

Imposing also $\det T = I$ yields the quantum groups $SL_{q,r}(N;\mathbf{R})$, $SL'_{q,r}(N;\mathbf{R})$ and $SU_{q,r}(N)$ respectively.

3 The universal enveloping algebra of $GL_{q,r}(N)$

We construct the universal enveloping algebra of $GL_{q,r}(N)$ as the algebra of regular functionals [8] on $GL_{q,r}(N)$: it is generated by the functionals $L^{\pm}, \varepsilon$ and Φ defined below.

Algebra structure

The $L^{\pm}$ linear functionals on $GL_{q,r}(N)$ are defined by their value on the matrix elements $T^A{}_B$:

$$L^{\pm A}{}_B(T^C{}_D) = (R^{\pm})^{AC}{}_{BD}, \tag{3.1}$$

$$L^{\pm A}{}_B(I) = \delta^A_B \tag{3.2}$$

with

$$(R^+)^{AC}{}_{BD} \equiv c^+ R^{CA}{}_{DB} \tag{3.3}$$

$$(R^-)^{AC}{}_{BD} \equiv c^-(R^{-1})^{AC}{}_{BD}, \tag{3.4}$$

where c^+, c^- are free parameters (see later).

To extend the definition (3.1) to the whole algebra $GL_{q,r}(N)$ we set

$$L^{\pm A}{}_B(ab) = L^{\pm A}{}_C(a) L^{\pm C}{}_B(b), \quad \forall a, b \in GL_{q,r}(N) \tag{3.5}$$

The commutations between $L^{\pm A}{}_B$ and $L^{\pm C}{}_D$ are induced by those between the T's:

$$R_{12} L_2^{\pm} L_1^{\pm} = L_1^{\pm} L_2^{\pm} R_{12} \tag{3.6}$$

$$R_{12}L_2^+L_1^- = L_1^-L_2^+R_{12}, \tag{3.7}$$

where as usual the product $L_2^{\pm}L_1^{\pm}$ is the convolution product $L_2^{\pm}L_1^{\pm} \equiv (L_2^{\pm} \otimes L_1^{\pm})\Delta$.

Note 1 : L^+ is upper triangular and L^- is lower triangular. Proof: apply L^+ and L^- to the T elements and use the upper and lower triangularity of R^+ and R^-, respectively.

A determinant can be defined for the matrix $L^{\pm A}{}_B$ as in (2.7) with $q \to q^{-1}$, $r \to r^{-1}$. Indeed the "RLL" relations are identical to the "RTT" with $R \to R^{-1}$ (which means $q \to q^{-1}$, $r \to r^{-1}$, cf. eq. (2.10)). Then , because of the upper or lower triangularity of L^+ and L^- respectively, we have

$$\det L^{\pm} = L^{\pm 1}{}_1 L^{\pm 2}{}_2 \cdots L^{\pm N}{}_N \tag{3.8}$$

A quantum inverse for $L^{\pm A}{}_B$ can be found, using an expression analogous to (2.25) with $q_{AB} \to q_{AB}^{-1}$.For this we need to introduce the element Φ defined by:

$$\Phi \det L^+ \det L^- = \det L^+ \det L^- \Phi = \varepsilon\ . \tag{3.9}$$

Then the quantum inverse of $L^{\pm A}{}_B$ is given by:

$$(L^{\pm A}{}_B)^{-1} = \Phi\ \det L^{\mp}\ \Pi^{(1,N)}_{BA}\ \ell^A_B \tag{3.10}$$

where ℓ^A_B is the quantum minor and $\Pi^{(1,N)}_{BA}$ is given in (2.26). Notice that $\Phi\ \det L^{\mp}$ is the inverse of $\det L^{\pm}$ because of property (3.23) below.

Coalgebra structure

The co-structures of the algebra generated by the functionals $L^{\pm}$, ε and Φ are defined by the duality (3.1):

$$\Delta'(L^{\pm A}{}_B)(a \otimes b) \equiv L^{\pm A}{}_B(ab) = L^{\pm A}{}_G(a) L^{\pm G}{}_B(b) \tag{3.11}$$

$$\varepsilon'(L^{\pm A}{}_B) \equiv L^{\pm A}{}_B(I) \tag{3.12}$$

$$\kappa'(L^{\pm A}{}_B)(a) \equiv L^{\pm A}{}_B(\kappa(a)) \tag{3.13}$$

so that

$$\Delta'(L^{\pm A}{}_B) = L^{\pm A}{}_G \otimes L^{\pm G}{}_B \tag{3.14}$$

$$\varepsilon'(L^{\pm A}{}_B) = \delta^A_B \tag{3.15}$$

$$\kappa'(L^{\pm A}{}_B) = L^{\pm A}{}_B \circ \kappa \tag{3.16}$$

$$\Delta'(\det L^{\pm}) = \det L^{\pm} \otimes \det L^{\pm}, \tag{3.17}$$

$$\Delta'(\Phi) = \Phi \otimes \Phi, \Delta'(\varepsilon) = \varepsilon \otimes \varepsilon \tag{3.18}$$

$$\varepsilon'(\det L^{\pm}) = 1,\ \varepsilon'(\Phi) = 1,\ \varepsilon'(\varepsilon) = 1 \tag{3.19}$$

$$\kappa'(\det L^{\pm}) = \Phi\ \det L^{\mp}, \tag{3.20}$$

$$\kappa'(\Phi) = \det L^+ \det L^-,\ \kappa'(\varepsilon) = \varepsilon \tag{3.21}$$

Note 2 : In (3.16) we have defined κ' using κ, we now prove that $\kappa'(L^{\pm A}{}_B) = (L^{\pm A}{}_B)^{-1}$ as defined in (3.10). This shows that $\kappa'(L^{\pm A}{}_B)$ is expressible by polynomials in $L^{\pm A}{}_B, \Phi$.

Proof : From $(L^{\pm})^{-1}L^{\pm} = \varepsilon$ we have $1 = [(L_1^{\pm})^{-1}L_1^{\pm}](T) = (L_1^{\pm})^{-1}(T_2)L_1^{\pm}(T_2) = (L_1^{\pm})^{-1}(T_2)R_{12}^{\pm}$ so that $(L_1^{\pm})^{-1}(T_2) = R_{12}^{\pm\,-1}$.
From $\kappa(T)T = 1$ we similarly have $[\kappa'(L_1^{\pm})](T_2) = R_{12}^{\pm\,-1}$ and therefore $\kappa'(L^{\pm A}{}_B) = (L^{\pm A}{}_B)^{-1}$.

Since κ' is an inner operation in the algebra generated by the functionals $L^{\pm A}{}_B$, ε and ϕ we conclude that these elements generate the Hopf algebra $U(gl_{q,r}(N))$ of the regular functionals on the quantum group $GL_{q,r}(N)$.

In the following we list some useful properties of the $L^{\pm}$ functionals.

Properties of $L^{\pm}$

i) From (3.6) and (3.7) we have

$$L^{\pm A}{}_A L^{\pm B}{}_B = L^{\pm B}{}_B L^{\pm A}{}_A \ ; \quad L^{+A}{}_A L^{-B}{}_B = L^{-B}{}_B L^{+A}{}_A \tag{3.22}$$

As a consequence:

$$\det L^+ \det L^- = \det L^- \det L^+. \tag{3.23}$$

ii) From (3.1) we deduce:

$$L^{\pm A}{}_B(\det T) = \delta^A_B (c^{\pm})^N r^{\pm 1} Q_A^{-1} \tag{3.24}$$

Proof: observe that $L^{\pm A}{}_B(\det T) = L^{\pm A}{}_B(T^1{}_1 T^2{}_2 \cdots T^N{}_N)$ since all the other permutations do not contribute, due to the structure of the $R^{\pm}$ matrix. Then it is easy to see that

$$L^{\pm A}{}_B(T^1{}_1 T^2{}_2 \cdots T^N{}_N) = \tag{3.25}$$

$$\delta^A_B (c^{\pm})^N (R^{\pm})^{A1}{}_{A1} (R^{\pm})^{A2}{}_{A2} \cdots (R^{\pm})^{AN}{}_{AN} = \delta^A_B (c^{\pm})^N r^{\pm 1} Q_A^{-1} \tag{3.26}$$

As a corollary, we have that :

$$L^{\pm A}{}_B(\Xi) = \delta^A_B (c^{\pm})^{-N} r^{\mp 1} Q_A \tag{3.27}$$

Proof: use (2.6) and (3.5) .

iii) From the expression (3.28) of $\det L^{\pm}$ we have

$$\det L^{\pm}(T^A{}_B) = \delta^A_B (c^{\pm})^N r^{\pm 1} Q_A \tag{3.28}$$

Proof:

$$\det L^{\pm}(T^A{}_B) = \delta^A_B (c^{\pm})^N (R^{\pm})^{1A}{}_{1A} \cdots (R^{\pm})^{NA}{}_{NA} = \delta^A_B (c^{\pm})^N r^{\pm 1} Q_A. \tag{3.29}$$

As a corollary we obtain

$$\det L^{\pm}(\det T) = \det R^{\pm} = (c^{\pm})^{N^2} r^{\pm N} \tag{3.30}$$

where the notation $\det R^{\pm}$ means the ordinary determinant of the square matrix $(R^{\pm})^{AB}{}_{CD}$ where rows and columns are respectively labelled by the combined indices AB and CD. Notice that

$$\det L^{\pm}(T^A{}_B) = L^{\pm A}{}_B(\det T) Q_A^2. \tag{3.31}$$

From (3.8) it is easy to see that $\det L^{\pm}(I) = 1$.

iv) Since the RLL relations are the same as the RTT relations with $q_{AB} \to (q_{AB})^{-1}$, $r \to r^{-1}$, we obtain a formula analogous to (2.16):

$$(\det L^{\pm}) L^{\pm A}{}_B = \frac{Q_B}{Q_A} L^{\pm A}{}_B (\det L^{\pm}). \tag{3.32}$$

We also have

$$(\det L^{\mp}) L^{\pm A}{}_B = \frac{Q_B}{Q_A} L^{\pm A}{}_B (\det L^{\mp}) \tag{3.33}$$

v) From (3.32) and (3.33) the following element:

$$\det L^{+} (\det L^{-})^{-1} = (\det L^{-})^{-1} \det L^{+} \tag{3.34}$$

is seen to be central. Notice that it is also group-like since

$$\Delta'(\det L^{\pm}) = \det L^{\pm} \otimes \det L^{\pm}. \tag{3.35}$$

In general even if $\det L^{+}(\det L^{-})^{-1}$ is central and group-like it is not equal to ε because

$$\det L^{+}(\det L^{-})^{-1}(T^A{}_B) = (c^{+})^N (c^{-})^{-N}\; r^2 \delta^A_B. \tag{3.36}$$

vi) The elements $L^{+A}{}_A L^{-A}{}_A$ (no sum on A) play a special role for particular values of the deformation parameters q_{AB}, r; if we set

$$L^{+A}{}_A L^{-A}{}_A \equiv \varepsilon_A \tag{3.37}$$

we leave as an exercise to deduce that (no sum on repeated indices):

$$\varepsilon_A(T^B{}_C) \equiv c^{+}c^{-}\delta^B_C\, \frac{q_{AB}^2}{r^2}, \quad \varepsilon_A(I) = 1, \quad \varepsilon_A(\Xi) = [\varepsilon_A(\det T)]^{-1} \tag{3.38}$$

$$\varepsilon_A(ab) = \varepsilon_A(a)\varepsilon_A(b), \quad a, b\; \in GL_{q,r}(N) \tag{3.39}$$

$$\kappa'(L^{\pm A}{}_A) = L^{\mp A}{}_A \varepsilon_A^{-1} \tag{3.40}$$

$$\varepsilon_A \varepsilon_B = \varepsilon_B \varepsilon_A, \qquad \varepsilon_A L^{\pm B}{}_B = L^{\pm B}{}_B \varepsilon_A \tag{3.41}$$

$$\text{det}L^+\text{det}L^- = \varepsilon_1 \cdots \varepsilon_N \ ; \tag{3.42}$$

$$\kappa'(\text{det}L^\pm) = \text{det}L^\mp(\varepsilon_1 \cdots \varepsilon_N)^{-1} = (\varepsilon_1 \cdots \varepsilon_N)^{-1}\text{det}L^\mp \tag{3.43}$$

Note 3: When det$T$ is central ($Q_A = 1$) we also have that $\det L^\pm$ is central (cf. (3.32) and (3.33)). If we set det$T = I$, then $L^{\pm A}{}_B(\text{det}T) = \delta^A_B(c^\pm)^N r^{\pm 1}$ must be equal to δ^A_B, or $c^\pm = r^{\mp\frac{1}{N}}\alpha^\pm$ with $(\alpha^\pm)^N = 1$. In this case $[\text{det}L^\pm](T^A{}_B) = \delta^A_B$ so that $\det L^\pm = \varepsilon$. Thus for $Q_A = 1, (c^\pm)^N r^{\pm 1} = 1$, the functionals $L^\pm$and ε generate the Hopf algebra $U(sl_{q,r}(N))$, and we have the simplified relations:

$$\text{det}L^+(\text{det}L^-)^{-1} = \varepsilon \tag{3.44}$$

$$[L^{\pm A}{}_B](\text{det}T) = \delta^A_B \quad \text{no sum on } A \tag{3.45}$$

$$[\text{det}L^\pm](T^A{}_B) = \delta^A_B \tag{3.46}$$

$$[\text{det}L^\pm](\text{det}T) = 1 \tag{3.47}$$

Note 4: When $q_{AB} = r$ we recover the standard uniparametric R matrix. We have also $Q_A = 1$ and

$$\forall\, A \ \ \varepsilon_A = \varepsilon \quad \text{i.e. } L^{+A}{}_A L^{-A}{}_A = L^{-A}{}_A L^{+A}{}_A = \varepsilon \tag{3.48}$$

In this case the Hopf algebra of functionals $U(gl_{q,r}(N))$ is equivalent to the algebra generated by the symbols $L^\pm, \Phi$ and ε modulo relations (3.6),(3.7) and (3.48).

Note 5: $GL_{q,r}(N)$ and $U(gl_{q,r}(N))$ are graded Hopf algebras: $T^A{}_B$ has grade $+1$, $\kappa(T^A{}_B)$ has grade -1, I has grade 0, $\det T$ has grade $+N$ etc., and similarly for $L^\pm$.

Conjugation

The *-conjugation on $GL_{q,r}(N)$ induces a *-conjugation on $U(gl_{q,r}(N))$ in two possible ways (we denote them as $*$ and $\sharp$-conjugations):

$$L^* \equiv \overline{L(\kappa(a^*))} \tag{3.49}$$

$$L^\sharp \equiv \overline{L(\kappa^{-1}(a^*))} \tag{3.50}$$

the overline being the usual complex conjugation. Both $*$ and $\sharp$ can be shown to satisfy all the properties of Hopf algebra involutions (2.35), (2.36). It is not difficult to determine their action on the basis elements $L^{\pm A}{}_B$. The three $GL_{q,r}(N)$ $*$-conjugations i), ii), iii) of the previous section induce respectively the following conjugations on the $L^{\pm A}{}_B$:

$$\begin{array}{ll} i) & (L^{\pm A}{}_B)^* = L^{\pm A}{}_B \\ ii) & (L^{\pm A}{}_B)^* = L^{\mp A'}{}_{B'} \\ iii) & (L^{\pm A}{}_B)^* = \kappa'^{-1}\,(L^{\mp B}{}_A) \end{array} \tag{3.51}$$

$$
\begin{array}{ll}
i) & (L^{\pm A}{}_B)^\sharp = \kappa'^2 \ (L^{\pm A}{}_B) \\
ii) & (L^{\pm A}{}_B)^\sharp = \kappa'^2 \ (L^{\mp A'}{}_{B'}) \\
iii) & (L^{\pm A}{}_B)^\sharp = \kappa' \ (L^{\mp B}{}_A)
\end{array} \tag{3.52}
$$

so that $(L^{\pm A}{}_B)^\sharp = \kappa'^2 \ [(L^{\pm A}{}_B)^*]$.

4 Differential calculus on $GL_{q,r}(N)$

The bicovariant differential calculus on the uniparametric q-groups of the A, B, C, D series can be formulated in terms of the corresponding R-matrix, or equivalently in terms of the $L^\pm$ functionals. This holds also for the multiparametric case. In fact all formulas are the same, modulo substituting the q parameter with r when it appears explicitly (typically as $\frac{1}{q-q^{-1}}$).

We briefly recall how to construct a bicovariant calculus. The general procedure can be found in ref. [15], or, in the notations we adopt here, in ref. [10]. It realizes the axiomatic construction of ref. [13].

The space of quantum 1-forms

As in the uniparametric case [15], the functionals

$$
f_{A_1}{}^{A_2 B_1}{}_{B_2} \equiv \kappa'(L^{+B_1}{}_{A_1}) L^{-A_2}{}_{B_2}. \tag{4.1}
$$

and the elements of $A = GL_{q,r}(N)$:

$$
M^{B_1}{}_{B_2 A_1}{}^{A_2} \equiv T^{B_1}{}_{A_1} \kappa(T^{A_2}{}_{B_2}). \tag{4.2}
$$

satisfy the following relations, where for simplicity we use the adjoint indices $i, j, k, ...$ with ${}^i = {}_A{}^B, \ {}_i = {}^A{}_B$:

$$
f^i{}_j(ab) = f^i{}_k(a) f^k{}_j(b) \tag{4.3}
$$
$$
f^i{}_j(I) = \delta^i_j \tag{4.4}
$$
$$
\Delta(M_j{}^i) = M_j{}^l \otimes M_l{}^i \tag{4.5}
$$
$$
\varepsilon(M_j{}^i) = \delta^i_j \tag{4.6}
$$
$$
M_i{}^j (a * f^i{}_k) = (f^j{}_i * a) M_k{}^i \tag{4.7}
$$

The star product between a functional on A and an element of A is defined as:

$$
\chi * a \equiv (id \otimes \chi)\Delta(a) \tag{4.8}
$$
$$
a * \chi \equiv (\chi \otimes id)\Delta(a), \qquad a \in A, \ \chi \in A' \tag{4.9}
$$

Relation (4.7) is easily checked for $a = T^A{}_B$ since in this case it is implied by the RTT relations; it holds for a generic a because of property (4.3).

The space of quantum one-forms is defined as a right A-module Γ freely generated by the *symbols* $\omega_{A_1}{}^{A_2}$:

$$\Gamma \equiv a^{A_1}{}_{A_2}\omega_{A_1}{}^{A_2}, \quad a^{A_1}{}_{A_2} \in A \tag{4.10}$$

Theorem (due to Woronowicz: see Theorem 2.5 in the last ref. of [13], p. 143): because of the properties (4.3)-(4.7), Γ becomes a bimodule over A with the following right multiplication:

$$\omega_{A_1}{}^{A_2} a = (f_{A_1}{}^{A_2 B_1}{}_{B_2} * a)\ \omega_{B_1}{}^{B_2}, \tag{4.11}$$

In particular:

$$\omega_{A_1}{}^{A_2} T^R{}_S = s(R^{-1})^{TB_1}{}_{CA_1}(R^{-1})^{A_2 C}{}_{B_2 S} T^R{}_T \omega_{B_1}{}^{B_2} \tag{4.12}$$

$$\omega_{A_1}{}^{A_2} \det T = s^N r^{-2} \frac{Q_{A_1}}{Q_{A_2}} (\det T) \omega_{A_1}{}^{A_2} \tag{4.13}$$

$$\omega_{A_1}{}^{A_2} \Xi = s^{-N} r^2 \frac{Q_{A_2}}{Q_{A_1}} (\Xi) \omega_{A_1}{}^{A_2} \tag{4.14}$$

Moreover we can define a left and a right action of $GL_{q,r}(N)$ on Γ:

$$\Delta_L : \Gamma \to A \otimes \Gamma \quad ; \quad \Delta_L(a^{A_1}{}_{A_2}\omega_{A_1}{}^{A_2}) \equiv \Delta(a^{A_1}{}_{A_2}) I \otimes \omega_{A_1}{}^{A_2} \tag{4.15}$$

$$\Delta_R : \Gamma \to \Gamma \otimes A \quad ; \quad \Delta_R(a^{A_1}{}_{A_2}\omega_{A_1}{}^{A_2}) \equiv \Delta(a^{A_1}{}_{A_2}) \omega_{B_1}{}^{B_2} \otimes M^{B_1}{}_{B_2 A_1}{}^{A_2}. \tag{4.16}$$

These actions commute

$$(id \otimes \Delta_R)\Delta_L = (\Delta_L \otimes id)\Delta_R \tag{4.17}$$

and give a bicovariant bimodule structure to Γ.

Note: : $\det T$ and Ξ commute with all the ω (and thus can be set to I) iff all Q_A are equal and for $s^N r^{-2} = 1$, or $s = r^{\frac{2}{N}}\alpha$ with $\alpha^N = 1$, which agrees with the condition found in Note 3 of previous section.

Exterior derivative

A derivative operator $d\ :\ A \longrightarrow \Gamma$ can be defined via the element $\tau \equiv \sum_A \omega_A{}^A \in \Gamma$. This element is easily shown to be left and right-invariant:

$$\Delta_L(\tau) = I \otimes \tau \quad ; \quad \Delta_R(\tau) = \tau \otimes I \tag{4.18}$$

and defines the derivative d by

$$da = \frac{1}{r - r^{-1}}[\tau a - a\tau]. \tag{4.19}$$

The factor $\frac{1}{r-r^{-1}}$ is necessary for a correct classical limit $r \to 1$. It is immediate to prove the Leibniz rule

$$d(ab) = (da)b + a(db), \quad \forall a, b \in A\ . \tag{4.20}$$

Another expression for the derivative is given by:

$$da = (\chi^{A_1}_{\ \ A_2} * a)\ \omega_{A_1}^{\ \ A_2} \tag{4.21}$$

where

$$\chi^A_{\ B} = \frac{1}{r - r^{-1}}[f_C^{\ CA}{}_B - \delta^A_B \varepsilon] \tag{4.22}$$

are the left-invariant vectors dual to the left-invariant 1-forms $\omega_{A_1}^{\ \ A_2}$. The equivalence of (4.19) and (4.21) can be shown by using the rule (4.11) for τa in the right-hand side of (4.19).

Using (4.21) we compute the exterior derivative on the basis elements of $GL_{q,r}(N)$, and on the q-determinant:

$$d\ T^A_{\ B} = \frac{1}{r - r^{-1}}[s\ (R^{-1})^{CR}_{\ \ \ ET}(R^{-1})^{TE}_{\ \ \ SB} T^A_{\ C} - \delta^S_R T^A_{\ B}]\ \omega_R^{\ S} \tag{4.23}$$

$$d\ \Xi = \frac{s^{-N} r^2 - 1}{r - r^{-1}}\ \Xi\ \tau \tag{4.24}$$

$$d\ \det T = \frac{s^N r^{-2} - 1}{r - r^{-1}}\ (\det T)\tau \tag{4.25}$$

The reader can verify via the Leibniz rule, and with the help of eq. (4.13), that $d[(\det T)\Xi] = d[\Xi(\det T)] = 0$.
Note: again $\det T = I = \Xi$ requires $s^N r^{-2} = 1$.

Every element ρ of Γ, which by definition is written in a unique way as $\rho = a^{A_1}{}_{A_2} \omega_{A_1}^{\ \ A_2}$, can also be written as

$$\rho = a_k db_k \tag{4.26}$$

for some a_k, b_k belonging to A. This can be proven directly by inverting the relations (4.23) and (4.24), after replacing the explicit values of the R^{-1} matrices. The result is an expression of the ω in terms of a linear combination of $\kappa(T)dT$, as in the classical case:

$$\begin{aligned} \omega_A^{\ A} &= \frac{r}{s(s - r^2 - r^4 + sr^4)}[(r^2 - s)\ \kappa(T^A_{\ B})dT^B_{\ A} + r^2(s-1)\ \kappa(T^C_{\ B})dT^B_{\ C}\theta^{CA} + \\ &\quad +(-r^2 - sr^2 + s + sr^4)\ \kappa(T^C_{\ B})dT^B_{\ C}\theta^{AC}], \qquad \text{no sum on A} && (4.27) \\ \omega_A^{\ B} &= -s^{-1}\frac{r}{q_{BA}}\kappa(T^B_{\ C})dT^C_{\ A}, \qquad A \neq B && (4.28) \end{aligned}$$

When $s = 1$, the classical limit $\omega_A^{\ B} \stackrel{q=r\to 1}{\longrightarrow} -\kappa(T^A_{\ C})dT^C_{\ B}$ reproduces the familiar formula $\omega = -g^{-1}dg$ for the left-invariant one-forms on the group manifold. More generally, for $s = r^\alpha, \alpha \in \mathbf{C}$ we have :

$$\omega_A^{\ A} \stackrel{r\to 1}{\longrightarrow} [\frac{2-\alpha}{2(\alpha-1)}\sum_B \kappa(T^A_{\ B})dT^B_{\ A} + \frac{\alpha}{2(\alpha-1)}\sum_B \sum_{C\neq A} \kappa(T^C_{\ B})dT^B_{\ C}], \quad \text{no sum on A,} \tag{4.29}$$

which shows that the inversion formula (4.27) diverges in the classical limit for $s = r$.

Due to the bi-invariance of τ the derivative operator d is compatible with the actions Δ_L and Δ_R:

$$\Delta_L(da) = (id \otimes d)\Delta(a) \tag{4.30}$$

$$\Delta_R(da) = (d \otimes id)\Delta(a), \tag{4.31}$$

These two properties express the fact that d commutes with the left and right action of the quantum group, as in the classical case.

We conclude that the exterior derivative (4.19) together with the properties (4.20), (4.30) , (4.31) and (4.26) realize the axioms of a first-order bicovariant differential calculus [13].

Tensor product

The tensor product between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes a\rho'$, $a(\rho \otimes \rho') = (a\rho) \otimes \rho'$ and $(\rho \otimes \rho')a = \rho \otimes (\rho' a)$. Left and right actions on $\Gamma \otimes \Gamma$ are defined by:

$$\Delta_L(\rho \otimes \rho') \equiv \rho_1\rho'_1 \otimes \rho_2 \otimes \rho'_2, \quad \Delta_L : \Gamma \otimes \Gamma \to A \otimes \Gamma \otimes \Gamma \tag{4.32}$$

$$\Delta_R(\rho \otimes \rho') \equiv \rho_1 \otimes \rho'_1 \otimes \rho_2\rho'_2, \quad \Delta_R : \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma \otimes A \tag{4.33}$$

where ρ_1, ρ_2, etc., are defined by

$$\Delta_L(\rho) = \rho_1 \otimes \rho_2, \quad \rho_1 \in A,\ \rho_2 \in \Gamma \tag{4.34}$$

$$\Delta_R(\rho) = \rho_1 \otimes \rho_2, \quad \rho_1 \in \Gamma,\ \rho_2 \in A. \tag{4.35}$$

The extension to $\Gamma^{\otimes n}$ is straightforward.

Exterior product

The exterior product of one-forms is consistently defined as:

$$\omega_{A_1}^{\ A_2} \wedge \omega_{D_1}^{\ D_2} \equiv \omega_{A_1}^{\ A_2} \otimes \omega_{D_1}^{\ D_2} - \Lambda_{A_1\ D_1}^{\ A_2\ D_2}\Big|^{C_1\ B_1}_{C_2\ B_2} \omega_{C_1}^{\ C_2} \otimes \omega_{B_1}^{\ B_2} \tag{4.36}$$

where the Λ tensor is given by:

$$\Lambda_{A_1\ D_1}^{\ A_2\ D_2}\Big|^{C_1\ B_1}_{C_2\ B_2} \equiv f_{A_1}^{\ A_2 B_1}{}_{B_2}(M^{C_1}_{\ C_2 D_1}{}^{D_2}) = \tag{4.37}$$

$$d^{F_2} d^{-1}_{C_2} R^{F_2 B_1}{}_{C_2 G_1} (R^{-1})^{C_1 G_1}{}_{E_1 A_1} (R^{-1})^{A_2 E_1}{}_{G_2 D_1} R^{G_2 D_2}{}_{B_2 F_2} \tag{4.38}$$

This matrix satisfies the characteristic equation:

$$(\Lambda + r^2 I)\ (\Lambda + r^{-2} I)\ (\Lambda - I) = 0 \tag{4.39}$$

due to the Hecke condition (2.11). For simplicity we will at times use the adjoint indices $i, j, k, \ldots$ with ${}^{i} = {}_{A}{}^{B}$, ${}_{i} = {}^{A}{}_{B}$. Then (4.39) applied to $\omega^r \otimes \omega^s$ yields:

$$(\Lambda^{ij}{}_{kl} + r^2 \delta^i_k \delta^j_l)(\Lambda^{kl}{}_{mn} + r^{-2} \delta^k_m \delta^l_n)(\Lambda^{mn}{}_{rs} - \delta^m_r \delta^n_s)\omega^r \otimes \omega^s =$$
$$(\Lambda^{ij}{}_{kl} + r^2 \delta^i_k \delta^j_l)(\Lambda^{kl}{}_{mn} + r^{-2} \delta^k_m \delta^l_n)\omega^m \wedge \omega^n = 0 \tag{4.40}$$

and it is easy to see that the last equality can be rewritten as

$$\omega^i \wedge \omega^j = -Z^{ij}{}_{kl} \omega^k \wedge \omega^l \tag{4.41}$$

$$Z^{ij}{}_{kl} \equiv \frac{1}{r^2 + r^{-2}} [\Lambda^{ij}{}_{kl} + \Lambda^{-1\,ij}{}_{kl}]. \tag{4.42}$$

Note: The inverse of Λ always exists, and is given by

$$(\Lambda^{-1})_{A_1}{}^{A_2}{}_{D_1}{}^{D_2}|^{B_1}{}_{B_2}{}^{C_1}{}_{C_2} = f_{D_1}{}^{D_2 B_1}{}_{B_2} (T^{A_2}{}_{C_2} \kappa^{-1}(T^{C_1}{}_{A_1})) =$$
$$R^{F_1 B_1}{}_{A_1 G_1} (R^{-1})^{A_2 G_1}{}_{E_2 D_1} (R^{-1})^{D_2 E_2}{}_{G_2 C_2} R^{G_2 C_1}{}_{B_2 F_1} (d^{-1})^{C_1} d_{F_1} \tag{4.43}$$

Exterior differential on $\Gamma^{\wedge n}$

Having the exterior product we can define the exterior differential on Γ:

$$d \;:\; \Gamma \to \Gamma \wedge \Gamma \tag{4.44}$$

$$d(a_k db_k) = da_k \wedge db_k \tag{4.45}$$

which can easily be extended to $\Gamma^{\wedge n}$ ($d : \Gamma^{\wedge n} \to \Gamma^{\wedge (n+1)}$, $\Gamma^{\wedge n}$ being defined as in the classical case but with the quantum permutation operator Λ [13]). The definition (4.45) is equivalent to the following :

$$d\theta \equiv \frac{1}{r - r^{-1}} [\tau \wedge \theta - (-1)^k \theta \wedge \tau], \tag{4.46}$$

where $\theta \in \Gamma^{\wedge k}$, and has the following properties:

$$d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta' \tag{4.47}$$

$$d(d\theta) = 0 \tag{4.48}$$

$$\Delta_L(d\theta) = (id \otimes d)\Delta_L(\theta) \tag{4.49}$$

$$\Delta_R(d\theta) = (d \otimes id)\Delta_R(\theta), \tag{4.50}$$

where $\theta \in \Gamma^{\wedge k}$, $\theta' \in \Gamma^{\wedge n}$.

Cartan-Maurer equations

The q-Cartan-Maurer equations are found by using (4.46) in computing $d\omega_{C_1}{}^{C_2}$:

$$d\omega_{C_1}{}^{C_2} = \frac{1}{r - r^{-1}} (\omega_B{}^B \wedge \omega_{C_1}{}^{C_2} + \omega_{C_1}{}^{C_2} \wedge \omega_B{}^B) \equiv -\frac{1}{2} C^{A_1}{}_{A_2}{}^{B_1}{}_{B_2}|_{C_1}{}^{C_2} \, \omega_{A_1}{}^{A_2} \wedge \omega_{B_1}{}^{B_2} \tag{4.51}$$

with:

$$C^{A_1\ B_1}_{\ A_2\ B_2}|^{C_2}_{C_1} = -\frac{2}{r-r^{-1}}[\delta^{A_1}_{C_1}\delta^{C_2}_{A_2}\delta^{B_1}_{B_2} - \frac{1}{r^2+r^{-2}}(\delta^{A_1}_{C_1}\delta^{C_2}_{A_2}\delta^{B_1}_{B_2} + \Lambda_{B\ C_1}^{\ B\ C_2}|^{A_1\ B_1}_{\ A_2\ B_2})] \quad (4.52)$$

To derive this formula we have used the flip operator Z on $\omega_B^{\ B} \wedge \omega_{C_1}^{\ C_2}$.

q-Lie algebra

Finally, we recall that the χ operators close on the q-Lie algebra:

$$\chi_i\chi_j - \Lambda^{kl}_{\ \ ij}\chi_k\chi_l = \mathbf{C}_{ij}^{\ \ k}\chi_k \quad (4.53)$$

where the q-structure constants are given by $\mathbf{C}_{jk}^{\ \ i} = \chi_k(M_j^{\ i})$ or explicitly:

$$\mathbf{C}^{A_1\ B_1}_{\ A_2\ B_2}|^{C_2}_{C_1} = \frac{1}{r-r^{-1}}[-\delta^{B_1}_{B_2}\delta^{A_1}_{C_1}\delta^{C_2}_{A_2} + \Lambda_{B\ C_1}^{\ B\ C_2}|^{A_1\ B_1}_{\ A_2\ B_2}]. \quad (4.54)$$

Comparing with (4.52) we find the relation between the structure constants C appearing in the Cartan-Maurer equations and the structure constants $\mathbf{C}$ of the q-Lie algebra of the tangent vectors in eq. (4.53):

$$C^{A_1\ B_1}_{\ A_2\ B_2}|^{C_2}_{C_1} = \frac{2}{r^2+r^{-2}}[-(r-r^{-1})\delta^{B_1}_{B_2}\delta^{A_1}_{C_1}\delta^{C_2}_{A_2} + \mathbf{C}^{A_1\ B_1}_{\ A_2\ B_2}|^{C_2}_{C_1}] \quad (4.55)$$

For $q, r \to 1$ the structure constants C and $\mathbf{C}$ coincide.

More in general the χ and f operators close on the algebra (4.53) and

$$\Lambda^{nm}_{\ \ ij}f^i_{\ p}f^j_{\ q} = f^n_{\ i}f^m_{\ j}\Lambda^{ij}_{\ \ pq} \quad (4.56)$$

$$\mathbf{C}_{mn}^{\ \ i}f^m_{\ j}f^n_{\ k} + f^i_{\ j}\chi_k = \Lambda^{pq}_{\ \ jk}\chi_p f^i_{\ q} + \mathbf{C}_{jk}^{\ \ l}f^i_{\ l} \quad (4.57)$$

$$\chi_k f^n_{\ l} = \Lambda^{ij}_{\ \ kl}f^n_{\ i}\chi_j, \quad (4.58)$$

This algebra is *sufficient* to define a bicovariant differential calculus on A (see e.g. [14]), and will be called "bicovariant algebra" in the sequel. By applying the relations defining the bicovariant algebra (called also "bicovariance conditions") to the element $M_r^{\ s}$ we can express them in the adjoint representation:

$$\mathbf{C}_{ri}^{\ \ n}\mathbf{C}_{nj}^{\ \ s} - \Lambda^{kl}_{\ \ ij}\mathbf{C}_{rk}^{\ \ n}\mathbf{C}_{nl}^{\ \ s} = \mathbf{C}_{ij}^{\ \ k}\mathbf{C}_{rk}^{\ \ s} \quad (q\text{-Jacobi identities}) \quad (4.59)$$

$$\Lambda^{nm}_{\ \ ij}\Lambda^{ik}_{\ \ rp}\Lambda^{js}_{\ \ kq} = \Lambda^{nk}_{\ \ ri}\Lambda^{ms}_{\ \ kj}\Lambda^{ij}_{\ \ pq} \quad (\text{Yang–Baxter}) \quad (4.60)$$

$$\mathbf{C}_{mn}^{\ \ i}\Lambda^{ml}_{\ \ rj}\Lambda^{ns}_{\ \ lk} + \Lambda^{il}_{\ \ rj}\mathbf{C}_{lk}^{\ \ s} = \Lambda^{pq}_{\ \ jk}\Lambda^{is}_{\ \ lq}\mathbf{C}_{rp}^{\ \ l} + \mathbf{C}_{jk}^{\ \ m}\Lambda^{is}_{\ \ rm} \quad (4.61)$$

$$\mathbf{C}_{rk}^{\ \ m}\Lambda^{ns}_{\ \ ml} = \Lambda^{ij}_{\ \ kl}\Lambda^{nm}_{\ \ ri}\mathbf{C}_{mj}^{\ \ s} \quad (4.62)$$

Using the definitions (4.22) and (4.1) it is not difficult to find the co-structures on the functionals χ and f:

$$\Delta'(\chi_i) = \chi_j \otimes f^j_{\ i} + \varepsilon \otimes \chi_i \quad (4.63)$$

$$\varepsilon'(\chi_i) = 0 \quad (4.64)$$

$$\kappa'(\chi_i) = -\chi_j\kappa'(f^j_{\ i}), \quad (4.65)$$

$$\Delta'(f^i{}_j) = f^i{}_k \otimes f^k{}_j \tag{4.66}$$
$$\varepsilon'(f^i{}_j) = \delta^i_j \tag{4.67}$$
$$\kappa'(f^k{}_j)f^j{}_i = \delta^k_i\varepsilon = f^k{}_j\kappa'(f^j{}_i) \tag{4.68}$$

Note that in the $r, q, s \to 1$ limit $f^i{}_j \to \delta^i_j\varepsilon$, i.e. $f^i{}_j$ becomes proportional to the identity functional and formula (4.11), becomes trivial, e.g. $\omega^i a = a\omega^i$ [use $\varepsilon * a = a$].

5 The quantum group $IGL_{q,r}(N)$

The q-inhomogeneous group $IGL_{q,r}(N)$ is freely generated by the non-commuting matrix elements $T^A{}_B$ $[A = (0, a); a : 1, ..N]$, the identity I and the inverse ξ of the q-determinant of T as defined in (2.7), modulo the relations:

$$T^0{}_a = 0 \tag{5.1}$$

and the relations:

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d = T^b{}_fT^a{}_eR^{ef}{}_{cd} \tag{5.2}$$
$$R^{ab}{}_{ef}T^e{}_cx^f = \frac{q_{0c}}{r}x^bT^a{}_c \tag{5.3}$$
$$R^{ab}{}_{ef}x^ex^f = rx^bx^a \tag{5.4}$$
$$q_{0a}T^a{}_cu = q_{0c}uT^a{}_c \tag{5.5}$$
$$q_{0a}x^au = ux^a \tag{5.6}$$

where $x^a \equiv T^a{}_0$ and $u \equiv T^0{}_0$.

It is not difficult to check that this algebra, endowed with the coproduct Δ, the counit ε and the coinverse κ defined by :

$$\Delta(T^A{}_B) = T^A{}_C \otimes T^C{}_A; \quad \varepsilon(T^A{}_B) = \delta^A_B; \quad \kappa(T) = T^{-1} \tag{5.7}$$
$$\Delta(\xi) = \xi \otimes \xi; \quad \varepsilon(\xi) = 1; \quad \kappa(\xi) = \det T \tag{5.8}$$
$$\Delta(I) = I \otimes I; \quad \varepsilon(I) = 1; \quad \kappa(I) = I \tag{5.9}$$

where the quantum inverse of $T^A{}_B$ is given by $(T^{-1})^A{}_B = \xi\ \Pi^{(0,N)}_{AB}\ t_B{}^A$ [see eq. (2.26): $t_B{}^A$ is the quantum minor], is a Hopf algebra. The proof goes as in uniparametric case (see the second ref. of [1]).

In the commutative limit it is the algebra of functions on $IGL(N)$ plus the dilatation $T^0{}_0$.

Relations (5.7)-(5.9) explicity read:

$$\Delta(T^a{}_b) = T^a{}_c \otimes T^c{}_b, \quad \Delta(I) = I \otimes I, \tag{5.10}$$
$$\Delta(x^a) = T^a{}_b \otimes x^b + x^a \otimes u \tag{5.11}$$
$$\Delta(u) = u \otimes u, \quad \Delta(\xi) = \xi \otimes \xi \tag{5.12}$$
$$\Delta(\det T^a{}_b) = \det T^a{}_b \otimes \det T^a{}_b \tag{5.13}$$

$$\varepsilon(T^a{}_b) = \delta^a_b, \quad \varepsilon(I) = 1, \tag{5.14}$$
$$\varepsilon(x^a) = 0 \tag{5.15}$$
$$\varepsilon(u) = \varepsilon(\xi) = 1 \tag{5.16}$$
$$\varepsilon(\det T^a{}_b) = 1 \tag{5.17}$$

$$\kappa(T^a{}_b) = (T^{-1})^a{}_b = \xi u \; \Pi^{(1,N)}_{ab} t_b{}^a \tag{5.18}$$
$$\kappa(I) = I, \tag{5.19}$$
$$\kappa(x^a) = -\kappa(T^a{}_b) x^b \kappa(u) \tag{5.20}$$
$$\kappa(u) = \det T^a{}_b \; \xi \tag{5.21}$$
$$\kappa(\xi) = u \; \det T^a{}_b, \quad \kappa(\det T^a{}_b) = \xi u \tag{5.22}$$

where for completeness we have included the expressions for the q-determinant of T. Note that $\kappa(u)u = I = u\kappa(u)$.

This procedure is very similar to that discussed for $GL_{q,r}(N+1)$ in Section 2: indeed both these Hopf algebrae are obtained from the algebra freely generated by $T^A{}_B, I, \Xi$ or ξ through the introduction of moduli relations i.e. as quotients of suitable two-sided ideals: the one generated by the RTT relations in the $GL_{q,r}(N+1)$ case, and the one generated by the (5.1)-(5.6) relations in the $IGL_{q,r}(N)$ case.

We now rederive the quantum group $IGL_{q,r}(N)$ as a quotient of $GL_{q,r}(N+1)$: all Hopf algebra properties of $IGL_{q,r}(N)$ will descend from those of $GL_{q,r}(N+1)$. The formalism employed will be useful in the next Section to deduce the differential calculus on $IGL_{q,r}(N)$ from the one on $GL_{q,r}(N+1)$.

We start from the observation that the R-matrix of $GL_{q,r}(N+1)$ can be written as (A=(0,a)):

$$R^{AB}{}_{CD} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & \frac{r}{q_{0b}}\delta^b_d & 0 & 0 \\ 0 & (r-r^{-1})\delta^a_d & \frac{q_{0a}}{r}\delta^a_c & 0 \\ 0 & 0 & 0 & R^{ab}{}_{cd} \end{pmatrix} \tag{5.23}$$

where $R^{ab}{}_{cd}$ is the R-matrix of $GL_{q,r}(N)$, and the indices AB are ordered as $00, 0b, a0, ab$.

It is apparent that the $GL_{q,r}(N+1)$ R matrix contains the information on $GL_{q,r}(N)$. We will show that it also contains the information about the quantum group $IGL_{q,r}(N)$.

In the index notation $A=(0,a)$ the RTT relations explicity read :

$$R^{ab}{}_{ef} T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd} \tag{5.24}$$
$$T^a{}_c T^b{}_0 = \frac{q_{ab}}{q_{c0}} T^b{}_0 T^a{}_c \tag{5.25}$$
$$T^a{}_0 T^b{}_d = \frac{q_{ab}}{q_{0d}} T^b{}_d T^a{}_0 + \frac{r}{q_{0d}}(r-r^{-1}) T^a{}_d T^b{}_0 \tag{5.26}$$

$$T^a{}_c T^0{}_d = \frac{q_{a0}}{q_{cd}} T^0{}_d T^a{}_c \tag{5.27}$$
$$T^0{}_c T^b{}_d = \frac{q_{0b}}{q_{cd}} T^b{}_d T^0{}_c + \frac{r}{q_{cd}}(r - r^{-1}) T^0{}_d T^b{}_c \tag{5.28}$$
$$T^a{}_0 T^b{}_0 = q_{ab} T^b{}_0 T^a{}_0 \tag{5.29}$$
$$T^0{}_c T^b{}_0 = \frac{q_{0b}}{q_{c0}} T^b{}_0 T^0{}_c \tag{5.30}$$
$$T^0{}_c T^0{}_d = q_{dc} T^0{}_d T^0{}_c \tag{5.31}$$
$$T^0{}_0 T^b{}_d = \frac{q_{0b}}{q_{0d}} T^b{}_d T^0{}_0 + \frac{r}{q_{0d}}(r - r^{-1}) T^0{}_d T^b{}_0 \tag{5.32}$$
$$T^0{}_0 T^b{}_0 = q_{0b} T^b{}_0 T^0{}_0 \tag{5.33}$$
$$T^0{}_0 T^0{}_d = q_{d0} T^0{}_d T^0{}_0 \tag{5.34}$$

where $a < b$ and $c < d$.

Consider now in $GL_{q,r}(N)$ the space H of all sums of monomials containing at least an element of the kind $T^0{}_a$ (i.e. H is the ideal in $GL_{q,r}(N+1)$ generated by the elements $T^0{}_a$ as we will see). Notice that $T^0{}_0 T^b{}_d - \frac{q_{0b}}{q_{0d}} T^b{}_d T^0{}_0$ is an element of H because of relation (5.32).

Proposition 1. H is the space of all sums of monomials of the kind $T^0{}_b a^b$ with $a^b \in GL_{q,r}(N+1)$.
Proof: every monomial in a generic element h of H contains at least a factor $T^0{}_a$. Use the RTT relations (5.27), (5.28), (5.30) and (5.34) to see that it can be shifted step by step to the left until we have a sum of monomials $h = T^0{}_b a^b$.

A similar proof holds for the

Proposition 2. H is the space of all sums of monomials of the kind $a^b T^0{}_b$ with $a^b \in GL_{q,r}(N+1)$.

We now prove that H is a Hopf ideal, i.e. an ideal in the $GL_{q,r}(N+1)$ algebra that is also compatible with the co-structures of $GL_{q,r}(N+1)$; this allows to structure $GL_{q,r}(N+1)/H$ as a Hopf algebra.

Theorem 1: The space H is a Hopf ideal in $GL_{q,r}(N+1)$ that is:

i) H is a two-sided ideal in $GL_{q,r}(N+1)$

ii) H is a co-ideal i.e.

$$\Delta_{N+1}(H) \subseteq H \otimes GL_{q,r}(N+1) + GL_{q,r}(N+1) \otimes H \; ; \quad \varepsilon_{N+1}(H) = 0 \tag{5.35}$$

iii) H is compatible with κ_{N+1} :

$$\kappa_{N+1}(H) \subseteq H \; . \tag{5.36}$$

Proof:

i) H is trivially a subalgebra of $GL_{q,r}(N+1)$. It is a right and left ideal since $\forall h \in H, \forall a \in GL_{q,r}(N+1)$ $ha \in H$ and $ah \in H$. This follows immediately from the definition of H as sums of monomials containing at least a factor $T^0{}_a$. H is the ideal in $GL_{q,r}(N+1)$ generated by the elements $T^0{}_a$.

ii) By virtue of *Proposition 1*, $\forall h \in H$:

$$\begin{aligned} \Delta_{N+1}(h) &= \Delta_{N+1}(T^0{}_b a) = (T^0{}_c \otimes T^c{}_b + T^0{}_0 \otimes T^0{}_b)[a_1 \otimes a_2] && (5.37)\\ &= \underbrace{T^0{}_c a_1}_{\in H} \otimes T^c{}_b a_2 + T^0{}_0 a_1 \otimes \underbrace{T^0{}_b a_2}_{\in H} && (5.38)\end{aligned}$$

with the symbolic notation $\Delta(a) \equiv a_1 \otimes a_2$. Moreover

$$\varepsilon_{N+1}(h) = \varepsilon_{N+1}(T^0{}_b a) = \varepsilon_{N+1}(T^0{}_b)\varepsilon_{N+1}(a) = 0 \; . \tag{5.39}$$

iii)

$$\kappa_{N+1}(T^0{}_b) = \Xi \; \Pi^{(0,N)}_{ob} \; t_b{}^0 \tag{5.40}$$

where $\Pi^{(0,N)}_{ob}$ is defined in (2.26) and it is easy to see that the quantum minor $t_b{}^0 \in H$ since it is the determinant of a matrix that has elements $T^0{}_a$ in the first row. By *Proposition 1* and *Proposition 2* we have

$$\kappa_{N+1}(h) = \kappa_{N+1}(T^0{}_b a) = \kappa_{N+1}(a)\kappa_{N+1}(T^0{}_b) \in H \; , \tag{5.41}$$

Q.E.D.

We now consider the quotient

$$\frac{GL_{q,r}(N+1)}{H} \; , \tag{5.42}$$

i.e. the space of all equivalence classes $\{a\}$, $a \in GL_{q,r}(N+1)$ where $\{a\}$ contains all those elements of $GL_{q,r}(N+1)$ that differ from a by an element of H; in particular $\{T^0{}_a\} = \{0\}$. Let us introduce the canonical projection

$$\begin{aligned} P \; : \quad GL_{q,r}(N+1) &\longrightarrow \; GL_{q,r}(N+1)/H && (5.43)\\ a &\longmapsto \; P(a) \equiv \{a\} && (5.44)\end{aligned}$$

Any element of $GL_{q,r}(N+1)/H$ is of the form $P(a)$. Also, $P(H) = 0$, i.e. $H = Ker(P)$.

Since H is a two-sided ideal, $GL_{q,r}(N+1)/H$ is an algebra with the following sum and products:

$$P(a) + P(b) \equiv P(a+b) \; ; \quad P(a)P(b) \equiv P(ab) \; ; \quad \mu P(a) \equiv P(\mu a), \quad \mu \in \mathbf{C} \tag{5.45}$$

We will use the following notation:

$$x^a \equiv P(T^a{}_0) \ ; \quad u \equiv P(T^0{}_0) \ ; \quad \xi \equiv P(\Xi) \tag{5.46}$$

and with abuse of symbols:

$$T^a{}_b \equiv P(T^a{}_b) \ ; \quad I \equiv P(I) \ ; \quad 0 \equiv P(0) \tag{5.47}$$

notice that $P(T^0{}_a) = P(0) = 0$. Using (5.45) it is easy to show that $T^a{}_b,\ x^a,\ u,\ \xi$ and I generate the algebra $GL_{q,r}(N+1)/H$. Moreover from the RTT relations $R_{12}T_1T_2 = T_2T_1R_{12}$ in $GL_{q,r}(N{+}1)$ we find the "$P(RTT)$" relations in $GL_{q,r}(N+1)/H$:

$$P(R_{12}T_1T_2) = P(T_2T_1R_{12}) \quad i.e. \quad R_{12}P(T_1)P(T_2) = P(T_2)P(T_1)R_{12} \tag{5.48}$$

that are explicity given in (5.2)-(5.6).

Since H is a Hopf ideal then $GL_{q,r}(N+1)/H$ is also a Hopf algebra with co-structures:

$$\Delta(P(a)) \equiv (P \otimes P)\Delta_{N+1}(a) \ ; \ \ \varepsilon(P(a)) \equiv \varepsilon_{N+1}(a) \ ; \ \ \kappa(P(a)) \equiv P(\kappa_{N+1}(a)) \tag{5.49}$$

Indeed (5.35) and (5.36) ensure that $\Delta,\ \varepsilon$, and κ are well defined. For example

$$(P \otimes P)\Delta_{N+1}(a) = (P \otimes P)\Delta_{N+1}(b) \quad \text{if} \quad P(a) = P(b) \ . \tag{5.50}$$

In order to prove the Hopf algebra axioms of Appendix A for $\Delta,\ \varepsilon,\ \kappa$ we just have to project those for $\Delta_{N+1},\ \varepsilon_{N+1},\ \kappa_{N+1}$. For example, the first axiom is proved by applying $P \otimes P \otimes P$ to $(\Delta_{N+1} \otimes id)\Delta_{N+1}(a) = (id \otimes \Delta_{N+1})\Delta_{N+1}(a)$. The other axioms are proved in a similar way.

Notice that on the generators $T^a{}_b,\ x^a,\ u,\ \xi$ and I the co-structures (5.49) act as in (5.7)-(5.9).

In conclusion: the elements $T^a{}_b,\ x^a,\ u,\ \xi$ and I generate the Hopf algebra $GL_{q,r}(N+1)/H$ and satisfy the "$P(RTT)$" commutation rules (5.2)-(5.6). The co-structures act on them exactly as the co-structures defined in (5.7)-(5.9). Therefore the quotient $GL_{q,r}(N+1)/H$ is the q-inhomogeneous group defined at the beginning of this section:

$$IGL_{q,r}(N) = \frac{GL_{q,r}(N+1)}{H} \ . \tag{5.51}$$

The canonical projection $P \ : \ GL_{q,r}(N+1) \rightarrow IGL_{q,r}(N)$ is an epimorphism between these two Hopf algebrae.

Note 1: From the commutations (5.5) - (5.6) we see that one can set $u = I$ only when $q_{0a} = 1$ for all a.

Note 2: $P(\det T^A{}_B) = u \ \det T^a{}_b$ is central in $IGL_{q,r}(N)$ only when $Q_A = 1$, *A=0,1,..N* (apply the projection P to eq. (2.16)). Note that here we have $Q_A \equiv \prod_{C=0}^N (\frac{q_{CA}}{r})$.

Theorem 2: The centrality of u is incompatible with the centrality of $\det T^a{}_b$.

Proof: Suppose that $q_{0a} = 1$ so that u is central. Then the centrality of $\det T^a{}_b$ is equivalent to the centrality of $P(\det T^A{}_B)$ and requires $Q_A = 1$ (previous Note); in particular $Q_0 \equiv \prod_{c=1}^{N} \frac{r}{q_{0c}} = 1$, which cannot be since for $q_{0a} = 1$ we find $Q_0 = r^N$. Q.E.D.

We end this Section by giving the commutations of $\det T^a{}_b$ and ξ with all the generators:

$$(\det T^c{}_d)T^a{}_b = \frac{Q_a}{Q_b} T^a{}_b(\det T^c{}_d), \qquad \zeta T^a{}_b = \frac{Q_b}{Q_a} T^a{}_b \zeta \tag{5.52}$$

$$(\det T^c{}_d)x^a = \frac{Q_a}{Q_0} x^a(\det T^c{}_d), \qquad \zeta x^a = \frac{Q_0}{Q_a} x^a \zeta \tag{5.53}$$

$$(\det T^c{}_d)u = u(\det T^c{}_d), \qquad \zeta u = u\zeta \tag{5.54}$$

where here $Q_a \equiv \prod_{c=1}^{N}(\frac{q_{ca}}{r})$ and ζ is the inverse of $\det T^c{}_d$, i.e. $\zeta \equiv u\xi$. We see that the commutations of $\det T^c{}_d$ with $T^a{}_b$ are the correct ones for $GL_{q,r}(N)$ (i.e. are identical to the ones deduced in Section 2). In the standard uniparametric case $Q_a = 1$, and the q-determinant $\det T^c{}_d$ becomes central (and likewise ζ), provided that also $Q_0 = 1$.

6 The differential calculus of $IGL_{q,r}(N)$

In this Section we present a bicovariant differential calculus on $IGL_{q,r}(N)$, based on the following set of functionals f and elements M :

$$\begin{aligned}
f_{a_1}{}^{a_2 b_1}{}_{b_2} &= \kappa'(L^{+b_1}{}_{a_1})L^{-a_2}{}_{b_2} \\
f_{a_1}{}^{a_2 0}{}_{b_2} &= \kappa'(L^{+0}{}_{a_1})L^{-a_2}{}_{b_2} \\
f_0{}^{a_2 b_1}{}_{b_2} &= \kappa'(L^{+b_1}{}_0)L^{-a_2}{}_{b_2} \equiv 0 \\
f_0{}^{a_2 0}{}_{b_2} &= \kappa'(L^{+0}{}_0)L^{-a_2}{}_{b_2}
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
M^{b_1}{}_{b_2 a_1}{}^{a_2} &= T^{b_1}{}_{a_1}\kappa(T^{a_2}{}_{b_2}) \\
M^{b_1}{}_{b_2 0}{}^{a_2} &= T^{b_1}{}_0\kappa(T^{a_2}{}_{b_2}) \\
M^0{}_{b_2 a_1}{}^{a_2} &= 0 \\
M^0{}_{b_2 0}{}^{a_2} &= T^0{}_0\kappa(T^{a_2}{}_{b_2})
\end{aligned} \tag{6.2}$$

The f in (6.1) are a subset of the f functionals of $GL_{q,r}(N+1)$, obtained by restricting the indices of $f^i{}_j$ to $i = ab$ and $i = 0b$. The third f is identically zero because of upper triangularity of L^+, i.e. $L^{+b_1}{}_0 = 0$.

The elements $M \in IGL_{q,r}(N)$ in (6.2) are obtained with the same restriction on the adjoint indices, and by projecting with P. The effect of the projection is to

replace the coinverse in $GL_{q,r}(N+1)$, i.e. κ_{N+1} , with the coinverse κ of $IGL_{q,r}(N)$ (see the last of (5.49)). The third element in (6.2) becomes zero because of P.

Theorem 1: the functionals in (6.1) vanish when applied to elements of $Ker(P) \subset IGL_{q,r}(N)$.

Proof: first one checks directly that the functionals (6.1) vanish when applied to $T^0{}_b$. This extends to any element of the form $T^0{}_b a$ $(a \in A)$, i.e. to any element of $Ker(P)$, because of the property (4.3) and the vanishing of the functionals in (6.7). Q.E.D.

The theorem states that the f functionals are well defined on the quotient $IGL_{q,r}(N) = GL_{q,r}(N+1)/Ker(P)$, in the sense that the "projected" functionals

$$\bar{f} : IGL_{q,r}(N) \to \mathbf{C}, \quad \bar{f}(P(a)) \equiv f(a) \ , \quad \forall a \in GL_{q,r}(N+1) \tag{6.3}$$

are well defined. Indeed if $P(a) = P(b)$, then $f(a) = f(b)$ because $f(Ker(P)) = 0$. This holds for any functional f vanishing on $Ker(P)$, not only for the $f^i{}_j$ functionals.

The product fg of two generic functionals vanishing on $KerP$ also vanishes on $KerP$, because $KerP$ is a co-ideal (see Theorem 1 in Section 5): $fg(KerP) = (f \otimes g)\Delta_{N+1}(KerP) = 0$. Therefore $\overline{fg}$ is well defined on $IGL_{q,r}(N)$, and

$$\overline{fg}[P(a)] = fg(a) = (f\otimes g)\Delta_{N+1}(a) = (\bar{f}P\otimes\bar{g}P)\Delta_{N+1}(a) = (\bar{f}\otimes\bar{g})\Delta(P(a)) \equiv \bar{f}\bar{g}[P(a)] \tag{6.4}$$

(use the first of (5.49)) so that the product of $\bar{f}$ and $\bar{g}$ involves the coproduct Δ of $IGL_{q,r}(N)$.

There is a natural way to introduce a coproduct on the $\bar{f}$'s :

$$\Delta' \bar{f}[P(a) \otimes P(b)] \equiv \bar{f}[P(a)P(b)] = \bar{f}[P(ab)] = f(ab) = \Delta'_{N+1} f(a \otimes b) \ . \tag{6.5}$$

It is also easy to show that

$$\Delta' \bar{f}^i{}_j = \bar{f}^i{}_k \otimes \bar{f}^k{}_j \quad \text{i.e.} \quad \bar{f}^i{}_j[P(a)P(b)] = \bar{f}^i{}_k[P(a)]\bar{f}^k{}_j[P(b)] \tag{6.6}$$

with i, j, k running over the restricted set of indices $ab, 0b$. Indeed due to

$$f_0{}^{A_2 b_1}{}_{B_2} \equiv 0, \quad f_{A_1}{}^{0B_1}{}_{b_2} \equiv 0 \tag{6.7}$$

(a consequence of upper and lower triangularity of L^+ and L^- respectively), formulae (4.66) and (4.3) involve only the $f^i{}_j$ listed in (6.1), which annihilate $KerP$. Then

$$\bar{f}^i{}_j[P(a)P(b)] = \bar{f}^i{}_j[P(ab)] = f^i{}_j(ab) = f^i{}_k(a) f^k{}_j(b) = \bar{f}^i{}_k[P(a)]\bar{f}^k{}_j[P(b)] \tag{6.8}$$

and (6.6) is proved.

With abuse of notations we will simply write f instead of $\bar{f}$. Then the f in (6.1) will be seen as functionals on $IGL_{q,r}(N)$.

Theorem 2: the right A-module ($A = IGL_{q,r}(N)$) Γ freely generated by $\omega^i \equiv \omega_{a_1}{}^{a_2}, \omega_0{}^{a_2}$ is a bicovariant bimodule over $IGL_{q,r}(N)$ with right multiplication:

$$\omega^i a = (f^i{}_j * a)\omega^j, \quad a \in IGL_{q,r}(N) \tag{6.9}$$

where the $f^i{}_j$ are given in (6.1), the $*$-product is computed with the co-product Δ of $IGL_{q,r}(N)$, and the left and right actions of $IGL_{q,r}(N)$ on Γ are given by

$$\Delta_L(a_i\omega^i) \equiv \Delta(a_i) I \otimes \omega^i \tag{6.10}$$

$$\Delta_R(a_i\omega^i) \equiv \Delta(a_i)\omega^j \otimes M_j{}^i \tag{6.11}$$

where the $M_j{}^i$ are given in (6.2).

Proof: we prove the theorem by showing that the functionals f and the elements M listed in (6.1) and (6.2) satisfy the properties (4.3)-(4.7) (cf. the theorem by Woronowicz discussed in the Section 4). It is straightforward to verify directly that the elements M in (6.2) do satisfy the properties (4.5) and (4.6). We have already shown that the functionals f in (6.1) satisfy (4.3), and property (4.4) obviously also holds for this subset.

Consider now the last property (4.7). We know that it holds for $GL_{q,r}(N+1)$. Take the free indices j and k as ab and $0b$, and apply the projection P on both members of the equation. It is an easy matter to show that only the f's in (6.1) and the M's in (6.2) enter the sums: this is due to the vanishing of some $P(M)$ and to (6.7). We still have to prove that the $*$ product in (4.7) can be computed via the coproduct Δ in $IGL_{q,r}(N)$. Consider the projection of property (4.7), written symbolically as:

$$P[M(f \otimes id)\Delta_{N+1}(a)] = P[(id \otimes f)\Delta_{N+1}(a)M] \ . \tag{6.12}$$

Now apply the definition (6.3) and the first of (5.49) to rewrite (6.12) as

$$P(M)(\bar{f} \otimes id)\Delta(P(a)) = (id \otimes \bar{f})\Delta(P(a))P(M) \ . \tag{6.13}$$

This projected equation then *becomes* property (4.7) for the $IGL_{q,r}(N)$ functionals f and adjoint elements M, with the correct coproduct Δ of $IGL_{q,r}(N)$. Q.E.D.

Using the general formula (6.9) we can deduce the ω, T commutations for $IGL_{q,r}(N)$:

$$\omega_{a_1}{}^{a_2} T^r{}_s = s(R^{-1})^{tb_1}{}_{ca_1}(R^{-1})^{a_2c}{}_{b_2s} T^r{}_t \omega_{b_1}{}^{b_2} \tag{6.14}$$

$$\omega_{a_1}{}^{a_2} x^r = s\frac{q_{0a_1}}{q_{0a_2}} x^r \omega_{a_1}{}^{a_2} - (r - r^{-1})\frac{sr}{q_{0a_2}} T^r{}_{a_1}\omega_0{}^{a_2} \tag{6.15}$$

$$\omega_{a_1}{}^{a_2} u = s\frac{q_{0a_1}}{q_{0a_2}} u \ \omega_{a_1}{}^{a_2} \tag{6.16}$$

$$\omega_{a_1}{}^{a_2} \det T^a{}_b = s^N r^{-2} \frac{Q_{a_1}}{Q_{a_2}} (\det T^a{}_b) \omega_{a_1}{}^{a_2} \tag{6.17}$$

$$\zeta \omega_{a_1}{}^{a_2} = s^N r^{-2} \frac{Q_{a_1}}{Q_{a_2}} \omega_{a_1}{}^{a_2} \zeta \tag{6.18}$$

$$\omega_0{}^{a_2} T^r{}_s = s \frac{r}{q_{0t}} (R^{-1})^{a_2 t}{}_{b_2 s} T^r{}_t \omega_0{}^{b_2} \tag{6.19}$$

$$\omega_0{}^{a_2} x^r = \frac{s}{q_{0a_2}} x^r \omega_0{}^{a_2} \tag{6.20}$$

$$\omega_0{}^{a_2} u = \frac{s}{q_{0a_2}} u \omega_0{}^{a_2} \tag{6.21}$$

$$\omega_0{}^{a_2} \det T^a{}_b = s^N r^{-2} \frac{Q_0}{Q_{a_2}} (\det T^a{}_b) \omega_0{}^{a_2} \tag{6.22}$$

$$\zeta \omega_0{}^{a_2} = s^N r^{-2} \frac{Q_0}{Q_{a_2}} \omega_0{}^{a_2} \zeta \tag{6.23}$$

Note: u commutes with all ω 's only if $q_{0a} = 1$ (cf. Note 1 of Section 5) and $s = 1$. This means that $u = I$ is consistent with the differential calculus on $IGL_{q_{0a}=1,r}(N)$ only if the additional condition $s = 1$ is satisfied.

The 1-form $\tau \equiv \sum_a \omega_a{}^a$ is bi-invariant, as one can check by using (6.10)-(6.11). Then an exterior derivative on $IGL_{q,r}(N)$ can be defined as in eq. (4.19), and satisfies the Leibniz rule. The alternative expression $da = (\chi_i * a)\omega^i$ (cf. (4.21)) continues to hold, where

$$\chi^a{}_b = \frac{1}{r - r^{-1}} [f_c{}^{ca}{}_b - \delta^a_b \varepsilon]$$
$$\chi^0{}_b = \frac{1}{r - r^{-1}} [f_c{}^{c0}{}_b] \tag{6.24}$$

are the left-invariant vectors dual to the left-invariant 1-forms $\omega_a{}^b$ and $\omega_0{}^b$. They are functionals on $IGL_{q,r}(N)$ and as a consequence of (6.6) we have

$$\Delta'(\chi^a{}_b) = \chi^c{}_d \otimes f_c{}^{da}{}_b + \varepsilon \otimes \chi^a{}_b \tag{6.25}$$
$$\Delta'(\chi^0{}_b) = \chi^c{}_d \otimes f_c{}^{d0}{}_b + \chi^0{}_d \otimes f_0{}^{d0}{}_b + \varepsilon \otimes \chi^0{}_b \tag{6.26}$$

The exterior derivative on the generators $T^a{}_b$ is given by formula (4.23) with lower case indices. For the other generators we find:

$$dx^a = -s \frac{r}{q_{0s}} T^a{}_s \omega_0{}^s + \frac{s-1}{r - r^{-1}} x^a \tau \tag{6.27}$$

$$du = \frac{s-1}{r - r^{-1}} u \tau \tag{6.28}$$

$$d\xi = \frac{s^{-N-1} r^2 - 1}{r - r^{-1}} \; \xi \; \tau \tag{6.29}$$

Moreover:

$$d(\det T^a{}_b) = \frac{s^N r^{-2} - 1}{r - r^{-1}} \, (\det T^a{}_b)\tau \tag{6.30}$$

$$d\zeta = \frac{s^{-N} r^2 - 1}{r - r^{-1}} \, \zeta\tau \tag{6.31}$$

($\zeta \equiv u\xi$). Again we find that $u = I$ implies $s = 1$, and $\det T^a{}_b = \zeta = I$ requires $s^N r^{-2} = 1$.

Every element ρ of Γ can be written as $\rho = a_k db_k$ for some a_k, b_k belonging to $IGL_{q,r}(N)$. In fact one has the same formula as in (4.27) for $\omega_m{}^n$, where all indices now are lower case. For $\omega_0{}^n$ we find:

$$\omega_0{}^n = -\frac{q_{0n}}{sr}[\kappa(T^n{}_a)dx^a + \kappa(x^n)du] \tag{6.32}$$

Finally, the two properties (4.30) and (4.31) hold also for $IGL_{q,r}(N)$, because of the bi-invariance of $\tau = \omega_a{}^a$. Thus all the axioms for a bicovariant first order differential calculus on $IGL_{q,r}(N)$ are satisfied.

The exterior product of left-invariant one-forms is defined as

$$\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl}\omega^k \otimes \omega^l \tag{6.33}$$

where

$$\Lambda^{ij}{}_{kl} = f^i{}_l(M_k{}^j) \tag{6.34}$$

This Λ tensor can in fact be obtained from the one of $GL_{q,r}(N+1)$ by restricting its indices to the subset $ab, 0b$. This is true because when $i, l = ab$ or $0b$ we have $f^i{}_l(KerP) = 0$ so that $f^i{}_l$ is well defined on $IGL_{q,r}(N)$, and we can write $f^i{}_l(M_k{}^j) = \bar{f}^i{}_l[P(M_k{}^j)]$ (see discussion after Theorem 1). The non-vanishing components of Λ read:

$$\Lambda^{a_2}_{a_1}{}^{d_2}_{d_1}|^{c_1}_{c_2}{}^{b_1}_{b_2} = d^{f_2} d^{-1}_{c_2} R^{f_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{e_1 a_1} (R^{-1})^{a_2 e_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 f_2} \tag{6.35}$$

$$\Lambda^{a_2}_{0}{}^{d_2}_{d_1}|^{c_1}_{c_2}{}^{0}_{b_2} = \frac{q_{0c_2}}{q_{0c_1}} (R^{-1})^{a_2 c_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 c_2} \tag{6.36}$$

$$\Lambda^{a_2}_{a_1}{}^{d_2}_{0}|^{c_1}_{c_2}{}^{0}_{b_2} = -(r - r^{-1})\frac{q_{0c_2}}{q_{0a_2}} \delta^{c_1}_{a_1} R^{a_2 d_2}{}_{b_2 c_2} \tag{6.37}$$

$$\Lambda^{a_2}_{a_1}{}^{d_2}_{0}|^{0}_{c_2}{}^{b_1}_{b_2} = \frac{q_{0a_1}}{q_{0a_2}} d^{f_2} d^{-1}_{c_2} R^{f_2 b_1}{}_{c_2 a_1} R^{a_2 d_2}{}_{b_2 f_2} \tag{6.38}$$

$$\Lambda^{a_2}_{0}{}^{d_2}_{0}|^{0}_{c_2}{}^{0}_{b_2} = \frac{q_{0c_2}}{q_{0a_2}} r^{-1} R^{a_2 d_2}{}_{b_2 c_2} \tag{6.39}$$

These components still satisfy the characteristic equation (4.39), because the Λ tensor of $GL_{q,r}(N+1)$ does satisfy this equation, and if the free adjoint indices are taken as ab, $0b$, only the components in (6.35)-(6.39) enter in (4.39). To prove this, consider $\Lambda^{ij}{}_{kl}$ with k, l of the type ab or $0b$ and observe that it vanishes unless also

i, j are of the type $ab, 0b$. (This can be checked directly via the formula (4.38)). Then equations (4.41) and (4.42) hold also for the ω's of $IGL_{q,r}(N)$.

Note that Λ^{-1} tensor of $IGL_{q,r}(N)$ can be obtained by specializing the indices in the Λ^{-1} tensor of $GL_{q,r}(N+1)$ given in (4.43), as we did for Λ. The reader can convince himself of this by i) observing that the $\Lambda^{-1}{}^{ij}{}_{kl}$ tensor of (4.43) also vanishes when $k, l = ab$ or $0b$ and i, j are not of the type $ab, 0b$; ii) considering the equation $\Lambda^{-1}{}^{ij}{}_{rs}\Lambda^{rs}{}_{kl} = \delta^i_k \delta^j_l$ for $k, l = ab$ or $0b$.

The exterior differential on $\Gamma^{\wedge n}$ can be defined as in Section 4 (eq. (4.46)), and satisfies all the properties (4.47)-(4.50). As for $GL_{q,r}(N+1)$ the last two hold because of the bi-invariance of τ.

The Cartan-Maurer equations are

$$d\omega^i = \frac{1}{r - r^{-1}}(\tau \wedge \omega^i + \omega^i \wedge \tau) = -\frac{1}{2} C_{jk}{}^i \omega^j \wedge \omega^k \tag{6.40}$$

with

$$C^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}\big|_{c_1}^{c_2} = \frac{2}{r^2 + r^{-2}}[-(r - r^{-1})\delta^{b_1}_{b_2}\delta^{a_1}_{c_1}\delta^{c_2}_{a_2} + \mathbf{C}^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}\big|_{c_1}^{c_2}] \tag{6.41}$$

$$C^{a_1}{}_{a_2}{}^{0}{}_{b_2}\big|_{0}^{c_2} = \frac{2}{r^2 + r^{-2}}\mathbf{C}^{a_1}{}_{a_2}{}^{0}{}_{b_2}\big|_{0}^{c_2} \tag{6.42}$$

$$C^{0}{}_{a_2}{}^{b_1}{}_{b_2}\big|_{0}^{c_2} = \frac{2}{r^2 + r^{-2}}[-(r - r^{-1})\delta^{b_1}_{b_2}\delta^{c_2}_{a_2} + \mathbf{C}^{0}{}_{a_2}{}^{b_1}{}_{b_2}\big|_{0}^{c_2}] \tag{6.43}$$

The structure constants $\mathbf{C}$ (appearing in the q-Lie algebra of $IGL_{q,r}(N)$, see later) are given by

$$\mathbf{C}^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}\big|_{d_1}^{d_2} = \frac{1}{r - r^{-1}}[-\delta^{b_1}_{b_2}\delta^{c_1}_{d_1}\delta^{d_2}_{c_2} + \Lambda_a{}^a{}_{d_1}{}^{d_2}\big|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}] \tag{6.44}$$

$$= \text{structure constants of } GL_{q,r}(N)$$

$$\mathbf{C}^{c_1}{}_{c_2}{}^{0}{}_{b_2}\big|_{0}^{d_2} = -\frac{q_{0c_2}}{q_{0c_1}} R^{c_1 d_2}{}_{b_2 c_2} \tag{6.45}$$

$$\mathbf{C}^{0}{}_{c_2}{}^{b_1}{}_{b_2}\big|_{0}^{d_2} = \frac{1}{r - r^{-1}}[-\delta^{b_1}_{b_2}\delta^{d_2}_{c_2} + d^{f_2} d^{-1}_{c_2} R^{f_2 b_1}{}_{c_2 a} R^{a d_2}{}_{b_2 f_2}] \tag{6.46}$$

We conclude this Section by checking that the functionals f and χ in (6.1) and (6.24) close on the algebra (4.53), (4.56)-(4.58), where the product of functionals is defined by the coproduct Δ in $IGL_{q,r}(N)$. This result is expected, since the functionals in (6.1) and (6.24) correspond to a bicovariant differential calculus on $IGL_{q,r}(N)$.

To prove this, we first note that in $GL_{q,r}(N+1)$ the subset in (6.1) and (6.24) closes by itself on the bicovariant algebra (4.53), (4.56)-(4.58). This is due to the particular index structure of the tensors $\mathbf{C}$ and Λ, and to the vanishing of the f components in (6.7). The nonvanishing components of $\mathbf{C}$ and Λ that enter the operatorial bicovariance conditions (where the free adjoint indices are taken as

$ab, 0b)$, are given in (6.44)-(6.46) and (6.35)-(6.38). Finally, we know that the f functionals vanish on $KerP$, and so do the χ functionals (as can be deduced from their definition in terms of the f functionals, eq. (4.22)). From the discussion after Theorem 1 it follows that they are well defined on $IGL_{q,r}(N)$, and that their products involve the $IGL_{q,r}(N)$ coproduct Δ.

Thus the relations (4.53), (4.56)-(4.58) hold for the functionals (6.1) and (6.24) on $IGL_{q,r}(N)$. They are the bicovariance conditions corresponding to a consistent differential calculus on $IGL_{q,r}(N)$.

Using the values of the Λ and $\mathbf{C}$ tensors in (6.35)-(6.38) and (6.44)-(6.46), we can explicitly write the "q-Lie algebra" of $IGL_{q,r}(N)$ as:

$$\chi^{c_1}_{\ c_2}\chi^{b_1}_{\ b_2} - \Lambda_{a_1\ d_1}^{\ a_2\ d_2}|^{c_1\ b_1}_{\ c_2\ b_2}\ \chi^{a_1}_{\ a_2}\chi^{d_1}_{\ d_2} = \frac{1}{r-r^{-1}}[-\delta^{b_1}_{b_2}\delta^{c_1}_{d_1}\delta^{d_2}_{c_2} + \Lambda_{a\ d_1}^{\ a\ d_2}|^{c_1\ b_1}_{\ c_2\ b_2}]\chi^{d_1}_{\ d_2} \tag{6.47}$$

$$\begin{aligned}
&\chi^{c_1}_{\ c_2}\chi^{0}_{\ b_2} + (r-r^{-1})\frac{q_{0c_2}}{q_{0a_2}}R^{a_2d_2}_{\ \ b_2c_2}\chi^{c_1}_{\ a_2}\chi^{0}_{\ d_2} - \\
&\qquad - \frac{q_{0c_2}}{q_{0c_1}}(R^{-1})^{a_2c_1}_{\ \ g_2d_1}R^{g_2d_2}_{\ \ b_2c_2}\chi^{0}_{\ a_2}\chi^{d_1}_{\ d_2} = -\frac{q_{0c_2}}{q_{0c_1}}R^{c_1d_2}_{\ \ b_2c_2}\ \chi^{0}_{\ d_2}
\end{aligned} \tag{6.48}$$

$$\begin{aligned}
&\chi^{0}_{\ c_2}\chi^{b_1}_{\ b_2} - \frac{q_{0a_1}}{q_{0a_2}}d^{f_2}d^{-1}_{c_2}R^{f_2b_1}_{\ \ c_2a_1}R^{a_2d_2}_{\ \ b_2f_2}\chi^{a_1}_{\ a_2}\chi^{0}_{\ d_2} = \\
&\qquad \frac{1}{r-r^{-1}}[-\delta^{b_1}_{b_2}\delta^{d_2}_{c_2} + d^{f_2}d^{-1}_{c_2}R^{f_2b_1}_{\ \ c_2a}R^{ad_2}_{\ \ b_2f_2}]\chi^{0}_{\ d_2}
\end{aligned} \tag{6.49}$$

$$\chi^{0}_{\ c_2}\chi^{0}_{\ b_2} - \frac{q_{0c_2}}{q_{0a_2}}r^{-1}\ R^{a_2d_2}_{\ \ b_2c_2}\ \chi^{0}_{\ a_2}\chi^{0}_{\ d_2} = 0 \tag{6.50}$$

where $\Lambda_{a_1\ d_1}^{\ a_2\ d_2}|^{c_1\ b_1}_{\ c_2\ b_2}$ is the braiding matrix of $GL_q(n)$, given in (6.35), so that the commutations in (6.47) are those of the q-subalgebra $GL_q(n)$. Note that the $r \to 1$ limit on the right hand sides of (6.47) and (6.49) is finite, since the terms in square parentheses are a (finite) series in $r - r^{-1}$ whose $0-th$ order part vanishes (see [10], eq. (5.55)).

7 The multiparametric quantum plane as a q-coset space

In this Section we derive the differential calculus on the quantum plane

$$\frac{IGL_{q,r}(N)}{GL_{q,r}(N)}, \tag{7.1}$$

i.e. the subalgebra generated by the coordinates x^a. The coordinates x^a satisfy the commutations (5.4):

$$R^{ab}_{\ \ ef}x^e x^f = r x^b x^a \tag{7.2}$$

The main difference with the more conventional approach to the quantum plane is that now the coordinates do not trivially commute with the $GL_{q,r}(N)$ q-group elements, but q-commute according to relations (5.3):

$$R^{ab}{}_{ef}T^e{}_c x^f = \frac{q_{0c}}{r} x^b T^a{}_c \tag{7.3}$$

Lemma: $\chi^b{}_c(a) = 0$ when a is a polynomial in x^a and u with all monomials containing at least one x^a. This is easily proved by observing that no tensor exists with the correct index structure. For $s = 1$ we can extend this lemma even to $u \cdots u$, since for example

$$\chi^b{}_c(u) = \frac{s-1}{r-r^{-1}}\delta^b_c \tag{7.4}$$

and using the coproduct rule (6.25) one finds that $\chi^b{}_c(u \cdots u)$ is always proportional to $s-1$.

Theorem: $\chi^b{}_c * a = 0$ when a is a polynomial in x^a and $s = 1$.

Proof: we have $\chi^b{}_c * a = (id \otimes \chi^b{}_c)(a_1 \otimes a_2) = a_1 \chi^b{}_c(a_2)$. We use here the standard notation $\Delta(a) \equiv a_1 \otimes a_2$. Since a_2 is a polynomial in x^a and u (use the coproduct rule (5.11)), and $\chi^b{}_c$ vanishes on such a polynomial when $s = 1$ (previous Lemma), the theorem is proved. Q.E.D.

Because of this theorem we will henceforth set $s = 1$: then we can write the exterior derivative of an element of the quantum plane as

$$da = (\chi_s * a)V^s \tag{7.5}$$

(with $\chi_s \equiv \chi^0{}_s$, $V^s \equiv \omega_0{}^s$), i.e. only in terms of the "q-vielbein" V^s.

The action and value of χ_s on the coordinates is easily computed, cf. the definition in (6.24):

$$\chi_s * x^a = -\frac{r}{q_{0s}} T^a{}_s, \quad \chi_s(x^a) = -\frac{r}{q_{0s}}\delta^a_s \tag{7.6}$$

so that the exterior derivative of x^a is:

$$dx^a = -\frac{r}{q_{0s}} T^a{}_s V^s \tag{7.7}$$

and gives the relation between the q-vielbein V^s and the differentials dx^a.

Theorem: The Leibniz rule for the "q-partial derivatives" χ_c is given by :

$$\chi_c * (ab) = (\chi_d * a) f^d{}_c * b + a\chi_c * b \tag{7.8}$$

where $f^d{}_c \equiv f_0{}^{d0}{}_c$.

Proof: compute the left-hand side using the coproduct (6.26):

$$\chi_c * (ab) = (id \otimes \chi_c)(a_1 b_1 \otimes a_2 b_2) = a_1 b_1 (\chi_d \otimes f^d{}_c + \varepsilon \otimes \chi_c)(a_2 \otimes b_2) = \chi_d(a_2) a_1 b_1 f^d{}_c(b_2) + a_1 \varepsilon(a_2) b_1 \chi_c(b_2) \tag{7.9}$$

This is easily seen to coincide with the right-hand side of (7.8), if one remembers that $a_1\varepsilon(a_2)=a$ in virtue of (A.5). Q.E.D.

The x^a and V^b q-commute as (cf. (6.20)):

$$V^a x^b = (q_{0a})^{-1} x^b V^a \tag{7.10}$$

and via eq. (7.7) and (7.3) we find the dx^a, x^bcommutations :

$$dx^a x^b = r^{-1} (R^{-1})^{ab}{}_{ef} x^f dx^e \tag{7.11}$$

After acting on this equation with d we obtain:

$$dx^a \wedge dx^b = -r^{-1} (R^{-1})^{ab}{}_{ef} dx^f \wedge dx^e \tag{7.12}$$

which reproduce the known commutations between the differentials of the quantum plane, cf. ref. [6].

The commutations between the partial derivatives are given in eq.(6.50).

As in [6], all the relations of this Section are covariant under the $GL_{q,r}(N)$ action:

$$x^a \longrightarrow T^a{}_b \otimes x^b \tag{7.13}$$

Note: the partial derivatives χ_c, and in general all the tangent vectors χ of this paper have "flat" indices. To compare them with the partial derivatives discussed in [6], which have "curved" indices, we need to define the functionals ∂_s:

$$\partial_s * a \equiv -\frac{q_{0a}}{r} (\chi_a * a) \kappa(T^a{}_s) \tag{7.14}$$

whose action and value on the coordinates is

$$\partial_s * x^a = \delta^a_s I, \quad \partial_s(x^a) = \varepsilon(\partial_s * x^a) = \delta^a_s \tag{7.15}$$

so that

$$da = (\partial_s * a)\ dx^s \tag{7.16}$$

which is equation (7.5) in "curved" indices (Note: ref. [6] adopts a definition of ∂_s such that $da = dx^s\ (\partial_s * a)$).

The results of this Section are applied to the multiparametric quantum plane $IGL_{qr}(2)/GL_{qr}(2)$ at the end of the Table. The usual relations of the uniparametric case [6] are recovered after setting $q = r$.

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A The Hopf algebra axioms

A Hopf algebra over the field K is a unital algebra over K endowed with the linear maps:

$$\Delta \ : \ \ A \to A \otimes A \tag{A.1}$$

$$\varepsilon \ : \ \ A \to K \tag{A.2}$$

$$\kappa \ : \ \ A \to A \tag{A.3}$$

satisfying the following properties $\forall a, b \in A$:

$$(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a) \tag{A.4}$$

$$(\varepsilon \otimes id)\Delta(a) = (id \otimes \varepsilon)\Delta(a) = a \tag{A.5}$$

$$m(\kappa \otimes id)\Delta(a) = m(id \otimes \kappa)\Delta(a) = \varepsilon(a)I \tag{A.6}$$

$$\Delta(ab) = \Delta(a)\Delta(b) \ ; \ \ \Delta(I) = I \otimes I \tag{A.7}$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \ ; \ \ \varepsilon(I) = 1 \tag{A.8}$$

where m is the multiplication map $m(a \otimes b) = ab$. From these axioms we deduce:

$$\kappa(ab) = \kappa(b)\kappa(a) \ ; \ \ \Delta[\kappa(a)] = \tau(\kappa \otimes \kappa)\Delta(a) \ ; \ \ \varepsilon[\kappa(a)] = \varepsilon(a) \ ; \ \ \kappa(I) = I \tag{A.9}$$

where $\tau(a \otimes b) = b \otimes a$ is the twist map.

Table

The quantum group $IGL_{q,r}(2)$ and its differential calculus

Parameters: $q(\equiv q_{12}), q_{01}, q_{02}, r$

R and D-matrices of $GL_{q,r}(2)$*:*

$$R^{ab}{}_{cd} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & \frac{r}{q} & 0 & 0 \\ 0 & r - r^{-1} & \frac{q}{r} & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad D^a{}_b = \begin{pmatrix} r & 0 \\ 0 & r^3 \end{pmatrix}$$

$T^A{}_B$ *(A,B=0,1,2): fundamental representation of* $IGL_{q,r}(2)$

$$T^A{}_B = \begin{pmatrix} u & x^1 & x^2 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix}$$

Determinant of $IGL_{q,r}(2)$ *and definition of* ξ

$$\det T^A{}_B = u \det T^a{}_b, \quad \text{where } \det T^a{}_b = \alpha\delta - \frac{r^2}{q}\beta\gamma$$

$$\xi \det T^A{}_B = \det T^A{}_B \xi = I$$

Basis elements generating $IGL_{q,r}(2)$

$$\alpha, \beta, \gamma, \delta, \; x^1, \; x^2, \; u, \; \xi$$

Commutations of the basis elements

$$\alpha\beta = \frac{r^2}{q}\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = \frac{r^2}{q}\delta\gamma$$

$$\beta\gamma = \frac{q^2}{r^2}\gamma\beta, \quad \alpha\delta - \delta\alpha = \frac{r}{q}(r - r^{-1})\beta\gamma,$$

$$\alpha x^1 = \frac{q_{01}}{r^2}x^1\alpha, \quad \beta x^1 = \frac{q_{02}}{r^2}x^1\beta,$$

$$\alpha x^2 = q\frac{q_{01}}{r^2}x^2\alpha, \quad \beta x^2 = q\frac{q_{02}}{r^2}x^2\beta,$$

$$\gamma x^1 = \frac{q_{01}}{q} x^1 \gamma - \frac{r}{q}(r - r^{-1}) \alpha x^2, \quad \delta x^1 = \frac{q_{02}}{q} x^1 \delta - \frac{r}{q}(r - r^{-1}) \beta x^2,$$

$$\gamma x^2 = \frac{q_{01}}{r^2} x^2 \gamma, \quad \delta x^2 = \frac{q_{02}}{r^2} x^2 \delta$$

$$x^1 x^2 = q x^2 x^1$$

$$T^a{}_b u = \frac{q_{0b}}{q_{0a}} u T^a{}_b, \quad x^a u = (q_{0a})^{-1} u x^a$$

$$(\det T^A{}_B) T^A{}_B = \frac{q_{0A} q_{1A} q_{2A}}{q_{0B} q_{1B} q_{2B}} T^A{}_B (\det T^A{}_B), \;\; q_{AA} \equiv r, \;\; q_{AB} \equiv \frac{r^2}{q_{BA}}$$

$$T^A{}_B \xi = \frac{q_{0A} q_{1A} q_{2A}}{q_{0B} q_{1B} q_{2B}} \xi T^A{}_B$$

Conditions for centrality of $\det T^A{}_B = u \det T^a{}_b$, $\det T^a{}_b$ *and* u

$$\begin{array}{lcl} \text{centrality of } u \det T^a{}_b & \Longleftrightarrow & q_{01} q_{02} = r^2, \;\; q_{01} = q \\ \text{centrality of } \det T^a{}_b & \Longleftrightarrow & q_{01} q_{02} = r^2, \;\; q = r \\ \text{centrality of } u & \Longleftrightarrow & q_{01} = q_{02} = 1 \end{array}$$

Inverse of $T^A{}_B$

$$(T^{-1})^A{}_B = \begin{pmatrix} \det T^a{}_b \xi & -(T^{-1})^a{}_b x^b \det T^a{}_b \xi \\ 0 & (T^{-1})^a{}_b \end{pmatrix}$$

$$(T^{-1})^a{}_b = \xi u \begin{pmatrix} \delta & -q^{-1} \beta \\ -q \gamma & \alpha \end{pmatrix}$$

Commutations of the left-invariant one-forms ω

Notations: $\omega^1 \equiv \omega_1{}^1, \omega^+ \equiv \omega_1{}^2, \omega^- \equiv \omega_2{}^1, \omega^2 \equiv \omega_2{}^2, \;\; V^1 \equiv \omega_0{}^1, V^2 \equiv \omega_0{}^2$

$$\omega^1 \wedge \omega^+ + \omega^+ \wedge \omega^1 = 0$$

$$\omega^1 \wedge \omega^- + \omega^- \wedge \omega^1 = 0$$

$$\omega^1 \wedge \omega^2 + \omega^2 \wedge \omega^1 = (1 - r^2) \omega^+ \wedge \omega^-$$

$$\omega^+ \wedge \omega^- + \omega^- \wedge \omega^+ = 0$$

$$\omega^2 \wedge \omega^+ + r^2 \omega^+ \wedge \omega^2 = r^2 (r^2 - 1) \omega^+ \wedge \omega^1$$

$$\omega^2 \wedge \omega^- + r^{-2} \omega^- \wedge \omega^2 = (1 - r^2) \omega^- \wedge \omega^1$$

$$\omega^2 \wedge \omega^2 = (r^2 - 1) \omega^+ \wedge \omega^-$$

$$\omega^1 \wedge \omega^1 = \omega^+ \wedge \omega^+ = \omega^- \wedge \omega^- = 0$$

$$\omega^1 \wedge V^1 + r^2 V^1 \wedge \omega^1 = 0$$

$$q^{-1}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^1 + V^1 \wedge \omega^+ = (1-r^{-2})\omega^1 \wedge V^2$$

$$\omega^- \wedge V^1 + \frac{r^2}{q}\frac{q_{02}}{q_{01}} V^1 \wedge \omega^- = 0$$

$$\omega^2 \wedge V^1 + V^1 \wedge \omega^2 = (1-r^{-2})q\frac{q_{01}}{q_{02}}\omega^- V^2$$

$$\omega^1 \wedge V^2 + V^2 \wedge \omega^1 = 0$$

$$q^{-1}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^2 + V^2 \wedge \omega^+ = 0$$

$$\frac{q}{r^2}\frac{q_{01}}{q_{02}}\omega^- \wedge V^2 + V^2 \wedge \omega^- = (1-r^2)V^1 \wedge \omega^1$$

$$\omega^2 \wedge V^2 + r^2 \wedge V^2 = (r^2-1)[(1-r^2)\omega^1 \wedge V^2 + \frac{r^2}{q}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^1]$$

Cartan-Maurer equations:

$$d\omega^1 + r\omega^+ \wedge \omega^- = 0$$

$$d\omega^+ + r\omega^+(-r^2\omega^1 + \omega^2) = 0$$

$$d\omega^- + r(-r^2\omega^1 + \omega^2)\omega^- = 0$$

$$d\omega^2 - r\omega^+ \wedge \omega^- = 0$$

$$dV^1 - \frac{q}{r}\frac{q_{01}}{q_{02}}\omega^- \wedge V^2 - r^{-1}\omega^1 \wedge V^1 = 0$$

$$dV^2 - \frac{r}{q}\frac{q_{02}}{q_{01}}\omega^+ \wedge V^1 - r^{-1}\omega^2 \wedge V^2 - (r-r^{-1})V^2 \wedge \omega^1 = 0$$

The q-Lie algebra:

Notations: $\chi^1 \equiv \chi^1{}_1, \chi^+ \equiv \chi^1{}_2, \chi^- \equiv \chi^2{}_1, \chi^2 \equiv \chi^2{}_, \; P^1 \equiv \chi^0{}_1, P^2 \equiv \chi^0{}_2$

$$\chi_1\chi_+ - \chi_+\chi_1 - (r^4 - r^2)\chi_2\chi_+ = r^3\chi_+$$

$$\chi_1\chi_- - \chi_-\chi_1 + (r^2-1)\chi_2\chi_- = -r\chi_-$$

$$\chi_1\chi_2 - \chi_2\chi_1 = 0$$

$$\chi_+\chi_- - \chi_-\chi_+ + (1-r^2)\chi_2\chi_1 - (1-r^2)\chi_2\chi_2 = r(\chi_1 - \chi_2)$$

$$\chi_+\chi_2 - r^2\chi_2\chi_+ = r\chi_+$$

$$\chi_-\chi_2 - r^{-2}\chi_2\chi_- = -r^{-1}\chi_-$$

$$r^2\chi_1 P_1 - P_1\chi_1 + (r^2-1)P_2\chi_- = -rP_1$$

$$q\frac{q_{01}}{q_{02}}\chi_+ P_1 - P_1\chi_+ - r^2(1-r^2)\chi_2 P_2 = r^3 P_2$$

$$\chi_- P_1 - \frac{q}{r^2}\frac{q_{01}}{q_{02}} P_1\chi_- = 0$$

$$\chi_2 P_1 - P_1\chi_2 = 0$$

$$\chi_1 P_2 - P_2\chi_1 + (r^2-1)\frac{q}{r^2}\frac{q_{01}}{q_{02}}\chi_+ P_1 = 0$$

$$\chi_+ P_2 - q^{-1}\frac{q_{02}}{q_{01}} P_2\chi_+ = 0$$

$$\frac{r^2}{q}\frac{q_{02}}{q_{01}}\chi_- P_2 - P_2\chi_- + (1-r^2)\chi_2 P_1 = -rP_1$$

$$r^2\chi_2 P_2 - P_2\chi_2 = -rP_2$$

$$P_1 P_2 - \frac{q}{r^2}\frac{q_{01}}{q_{02}} P_2 P_1 = 0$$

The exterior derivative of the basis elements

$$\begin{aligned}
&d\alpha = \tfrac{s-r^2}{r^3-q}\alpha\omega^1 - s\tfrac{r}{q_{12}}\beta\omega^+ + \tfrac{s-1}{r-r^{-1}}\alpha\omega^2\\
&d\beta = \tfrac{-r^2+s(1-r^2+r^4)}{r^3-r}\beta\omega^1 - s\tfrac{q_{12}}{r}\alpha\omega^- + \tfrac{s-r^2}{r^3-r}\beta\omega^2\\
&d\gamma = \tfrac{s-r^2}{r^3-r}\gamma\omega^1 - s\tfrac{r}{q_{12}}\delta\omega^+ + \tfrac{s-1}{r-r^{-1}}\gamma\omega^2\\
&d\delta = \tfrac{-r^2+s(1-r^2+r^4)}{r^3-r}\delta\omega^1 - s\tfrac{q_{12}}{r}\gamma\omega^- + \tfrac{s-r^2}{r^3-r}\delta\omega^2\\
&dx^1 = -\tfrac{sr}{q_{01}}\alpha V^1 - \tfrac{sr}{q_{02}}\beta V^2 + \tfrac{s-1}{r-r^{-1}}x^1\tau\\
&dx^2 = -\tfrac{sr}{q_{01}}\gamma V^1 - \tfrac{sr}{q_{02}}\delta V^2 + \tfrac{s-1}{r-r^{-1}}x^2\tau\\
&du = \tfrac{s-1}{r-r^{-1}}u\tau\\
&d\xi = \tfrac{r^2 s^{-N-1}-1}{r-r^{-1}}\xi\tau, \quad d\zeta = \tfrac{r^2 s^{-N}-1}{r-r^{-1}}\zeta\tau\\
&d(\det T^A{}_B) = \tfrac{r^{-2}s^{N+1}-1}{r-r^{-1}}(\det T^A{}_B)\tau, \quad d(\det T^a{}_b) = \tfrac{r^{-2}s^N-1}{r-r^{-1}}(\det T^A{}_B)\tau
\end{aligned}$$

The ω^i in terms of the exterior derivatives on $\alpha, \beta, \gamma, \delta, x^1, x^2, u$:

$$\begin{aligned}
&\omega^1 = \tfrac{r}{s(-r^2-r^4+s+sr^4)}[(r^2-s)(\kappa(\alpha)da + \kappa(\beta)d\gamma) + r^2(s-1)(\kappa(\gamma)d\beta + \kappa(\delta)d\delta)]\\
&\omega^+ = -\tfrac{1}{s}\tfrac{q_{12}}{r}[\kappa(\gamma)d\alpha + \kappa(\delta)d\gamma]\\
&\omega^- = -\tfrac{1}{s}\tfrac{r}{q_{12}}[\kappa(\alpha)d\beta + \kappa(\beta)d\delta]\\
&\omega^2 = \tfrac{r}{s(-r^2-r^4+s+sr^4)}[(s-r^2-sr^2+sr^4)(\kappa(\alpha)d\alpha + \kappa(\beta)d\gamma) + (r^2-s)(\kappa(\gamma)d\beta + \kappa(\delta)d\delta)]\\
&V^1 = -\tfrac{q_{01}}{sr}[\kappa(\alpha)dx^1 + \kappa(\beta)dx^2 + \kappa(x^1)du]\\
&V^2 = -\tfrac{q_{02}}{sr}[\kappa(\gamma)dx^1 + \kappa(\delta)dx^2 + \kappa(x^2)du]
\end{aligned}$$

The multiparametric quantum plane $IGL_{qr}(2)/GL_{qr}(2)$

$$x^1x^2 = qx^2x^1$$

$$\begin{aligned}
&dx^1x^1 = r^{-2}x^1dx^1 \\
&dx^1\ x^2 = \tfrac{q}{r^2}x^2dx^1 \\
&dx^2\ x^1 = (r^{-2}-1)x^2dx^1 + q^{-1}x^1dx^2 \\
&dx^2x^2 = r^{-2}x^2dx^2
\end{aligned}$$

$$dx^1 \wedge dx^2 = -\tfrac{q}{r^2}dx^2 \wedge dx^1$$

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